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ON THE DETERMINANT OF (0,1) MATRICES

by

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I. Introduction

a) In the present paper we consider $n\times n$ matrices with elements 0,1 and our purpose is to investigate the number of all non-singular ones. We shall prove that the singular matrices form a negligible percent asymptotically. More precisely, we shall prove the following

THEOREM

Let A_n denote the number of $n\times n$ matrices with elements 0, 1 *having determinant* 0, *then*

$$\lim_{n=+\infty} \frac{A_n}{2^{n^2}} = 0.$$

b) In other words let us choose at random a matrix from the set of $n\times n$ (0, 1) matrices such that all matrices have the same probability (2^{-n^2}). If a_n means the probability of the event that the determinant of the chosen matrix equals 0, then $\lim\limits_{n=+\infty} a_n=0$. It is easy to see that the following fact is equivalent to our theorem:

If $\varepsilon_{i,j}$ are independent random variables which take the values 0 and 1 with probabilities $\frac{1}{2}, \frac{1}{2}$ and

$$p_n = \mathbf{P}\left(\begin{vmatrix} \varepsilon_{1,1} & \varepsilon_{1,2} \cdots \varepsilon_{1,n} \\ \varepsilon_{2,1} & \varepsilon_{2,2} \cdots \varepsilon_{2,n} \\ \cdots & \cdots \\ \varepsilon_{n,1} & \varepsilon_{n,2} \cdots \varepsilon_{n,n} \end{vmatrix} = 0\right)$$

then

$$\lim_{n=+\infty} p_n = 0.$$

We shall use all versions at the same time. In the section **VI.** we deal with a generalization of this problem in the case of infinite matrices.

c) The proof goes as follows: We show that the probability of the event, that the rank of an $n\times n$ (0, 1) matrix is $k+2$, where k denotes the rank of the $(n-1)\times\times(n-1)$ matrix, consisting of its first $n-1$ rows and columns, or is equal to n, tends to 1 if $n\to\infty$.

Using this fact we prove that

$$\liminf_{n=+\infty} \frac{A_n}{2^{n^2}} = 0.$$

Having proved this, we prove the convergence of the sequence $A_n/2^{n^2}$.

Before the proof of the theorem we give some definitions and lemmas.

II. Definitions and Lemmas

a) Let $A_{n,k}$ denote the number of $n \times n$ (0, 1) matrices whose rank is equal to k. Clearly

$$A_n = \sum_{k=1}^{n-1} A_{n,k} = 2^{n^2} - A_{n,n}.$$

Then we have to prove that

$$\lim_{n=+\infty} \frac{A_{n,n}}{2^{n^2}} = 1.$$

First we give a known lemma.

LEMMA 1. *Let $a_1, a_2, \dots, a_n$ be real numbers different from 0 and c an arbitrary real number, then at most $\binom{n}{\left[\frac{n}{2}\right]}$ among the sums $\sum_{i=1}^{n} \varepsilon_i a_i$ (ε_i is equal to 0 or 1) are equal to c.*

PROOF. Let us consider instead of the numbers $\sum_{i=1}^{n} \varepsilon_i a_i$ the sums $2 \cdot \sum_{i=1}^{n} \varepsilon_i a_i - \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \varphi_i a_i$, where $\varphi_i = 2\varepsilon_i - 1$, then φ_i is equal to 1 or -1 if ε_i is equal to 1 or 0, respectively. The sum $\sum_{i=1}^{n} \varepsilon_i a_i$ equals c if the sum $\sum_{i=1}^{n} \varphi_i a_i$ equals $d = 2c - \sum_{i=1}^{n} a_i$. Then we can reformulate the lemma so that the numbers ε_i are equal to 1 or -1. In this case we can suppose without violating the generality, that the numbers $a_1, a_2, \dots, a_n$ are all positive.

Then it is enough to prove the following: if $a_1, a_2, \dots, a_n$ are positive numbers and d is an arbitrary real number, then at most $\binom{n}{\left[\frac{n}{2}\right]}$ among the numbers $\sum_{i=1}^{n} \varepsilon_i a_i$ (ε_i equals 1 or -1) are equal to d.

Let us correspond for every sum $\sum_{i=1}^{n} \varepsilon_i a_i$ the set of those natural numbers i for which $\varepsilon_i = 1$ holds. If for two different sums $\sum_{i=1}^{n} \varepsilon_i a_i = \sum_{i=1}^{n} \varepsilon_i' a_i$, then the corresponding sets of the two sums cannot contain each other.

The Sperner-theorem implies that the number of sums equal to any constant is at most $\binom{n}{\left[\frac{n}{2}\right]}$.

Clearly we can formulate the lemma as follows: if $a_1, a_2, \dots, a_m$ are real numbers, among which n are different from 0 and c is an arbitrary real number then among the numbers $\sum_{i=1}^{n} \varepsilon_i a_i$ (ε_i equals 0 or 1) at most $\binom{n}{\left[\frac{n}{2}\right]} 2^{m-n} < \frac{2^m}{\sqrt{n}}$ are equal to c.

b)

DEFINITIONS.

A system of k linearly independent row (resp. column) vectors of a matrix of rank k is called a row (resp. column) basis of the matrix.

We shall use that any row (resp. column) vector is a uniquely determined linear combination of the vectors of any fixed row (resp. column) basis.

1) *The degree of a row (resp. column) vector with respect to a given row (resp. column) basis, is the number of those elements of the row (resp. column) basis, which have coefficients different from* 0 *in the above mentioned linear combination.*

2) *The degree of a row (resp. column) vector is the largest one among the degrees of this row (resp. column) vector with respect to all possible row (resp. column) basises.*

3) *The row (resp. column) degree of a matrix is the largest one among the degrees of its row (resp. column) vectors.*

LEMMA 2. *If the row-degree of an* $m \times n$ (0, 1) *matrix is l and its rank is k, then we can add to the matrix a column vector (with components* 0, 1*) so that the rank of the obtained* $m \times (n+1)$ *matrix is k again, at most* $\frac{2 \cdot 2^m}{\sqrt{l}}$ *different ways.*

PROOF. For the sake of simplicity let us suppose that the first k row vectors form the basis, with respect to which the degree of the t-th row vector is equal to l.

Let us denote the i-th row vector by $\mathbf{a}_i$, the j-th column vector by $\mathbf{b}_j$ and the additional (the $(n+1)$-th) column vector by $\mathbf{b}_{n+1}$, i. e.

$$\mathbf{a}_i = (a_{i,1}; a_{i,2}; \dots; a_{i,n}),$$

$$\mathbf{b}_j = \begin{pmatrix} b_{1,j} \\ b_{2,j} \\ \vdots \\ b_{m,j} \end{pmatrix}, \quad \mathbf{b}_{n+1} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

The row vectors of the enlarged matrix are

$$\mathbf{a}'_i = (a_{i,1}; a_{i,2}; \dots; a_{i,n}; b_i).$$

So we have $\mathbf{a}_t = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \dots + c_k\mathbf{a}_k$ where among the constants c_i l are different from 0.

If the degree of the new $(m \times (n+1))$ matrix is also k then (because the maximal numbers of linearly independent row and column vectors are equal to each other and clearly $\mathbf{a}'_1, \mathbf{a}'_2, \dots, \mathbf{a}'_k$ are also linearly independent)

$$\mathbf{a}'_t = c_1\mathbf{a}'_1 + c_2\mathbf{a}'_2 + \dots + c_k\mathbf{a}'_k$$

hence

$$b_t = c_1 b_1 + c_2 b_2 + \dots + c_k b_k.$$

But b_i is equal to 0 or 1 and among the numbers c_i l are different from 0, so by Lemma 1 we can choose the vector $(b_1, b_2, \ldots, b_k)$ at most $\frac{2^k}{\sqrt{l}}$ different ways such that $b_t=0$ holds; similarly we can choose $(b_1, b_2, \ldots, b_k)$ at most $\frac{2^k}{\sqrt{l}}$ different ways such that $b_t=1$ holds. That is, we have at most $\frac{2\cdot 2^m}{\sqrt{l}}$ possibilities to choose the vector $\mathbf{b}_{n+1}$. Q. e.d.

Similarly, if the column-degree of a matrix is l, then we can construct to the matrix a row vector at most $\frac{2\cdot 2^n}{\sqrt{l}}$ different ways such that the maximal numbers of linearly independent vectors of both matrices are equal to each other.

c)

LEMMA 3. *By k m-dimensional vectors (with elements* 0, 1*) we can construct at most* 2^{2^k} *different vectors (with components* 0, 1*) with linear combinations.*

PROOF. Let us consider a $k\times m$ matrix with row vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$. It contains at most 2^k different column vectors (because it has only 0 or 1 components). If the i_1-th, i_2-th, ..., i_t-th column vectors are the different ones ($t\leqq 2^k$), so any of the others is equal to one of these, then in the linear combinations of the row vectors the i_1-th, i_2-th, ..., i_t-th components can arbitrarily vary. Then among the linear combinations, whose components are 0, 1, at most $2^t\leqq 2^{2^k}$ can be different.
Q. e. d.

LEMMA 4. *There exists a natural number m_0 so that the number of those $m\times n$ (0, 1) matrices whose row-degree is at most* $\log m$ *but not equal to* 1, *is less than* $2^{n(m-1)}\cdot 2^{m^{4/5}}$ *if $m>m_0$.*

PROOF. Let us denote by D_l the number of those $m\times n$ (0, 1) matrices, whose row-degree is l and by $D_{l,i}$; the number of those (0, 1) matrices in which the i-th row vector has degree l. Then

$$D_l \leqq \sum_{i=1}^{m} D_{l,i} = m\cdot D_{l,m}$$

(because evidently $D_{l,1}=D_{l,2}=\ldots=D_{l,m}$).

We shall prove that

$$D_{l,m}<m^l\cdot 2^{2^l}\cdot 2^{n(m-1)} \qquad (l\geqq 2),$$

what proves our Lemma because the number of those matrices whose row-degree is at most $\log m$ but is not equal to 1 is

$$\sum_{l=2}^{[\log m]} D_l \leqq 2^{n(m-1)} \sum_{l=2}^{[\log m]} m\cdot m^l\cdot 2^{2^l} < 2^{n(m-1)}\cdot \log m\cdot m\cdot m^{\log m}\cdot 2^{2^{\log m}} < 2^{n(m-1)}\cdot 2^{m^{4/5}}$$

if $m>m_0$ for some suitable natural number m_0.

If we fill in the first $m-1$ rows of the matrix arbitrarily (it can be done by $2^{n(m-1)}$ different ways) we can construct the last row using a row-basis consisting of the first $m-1$ rows by a linear combination (because $l \geqq 2$) but actually we use only l rows of the row-basis, because the coefficients of the other rows are equal to 0. We have $\binom{m-1}{l} < m^l$ possibilities to choose the l vectors and by l vectors we can construct at most 2^{2^l} vectors as linear combinations according to Lemma3, that is

$$D_{l,m} < m^l \cdot 2^{2^l} \cdot 2^{n(m-1)} \qquad (l \geqq 2).$$

Q. e. d.

Similarly the number of $m \times n$ (0, 1) matrices whose column-degree is at most $\log n$ but is not equal to 1, is less than $2^{m(n-1)} \cdot 2^{n^{4/5}}$, if $n > m_0$.

If the row-(resp. column) degree of an $m \times n$ matrix is equal to 1, then we have two possibilities: either there are two rows(resp. columns) which are equivalent (the number of such matrices is less than $m^2 \cdot 2^{(m-1)n}$ (resp. $n^2 \cdot 2^{m(n-1)}$), or the rank of the matrix is m (resp. n) — these are the good cases for us.

d) Let us consider an $n \times n$ (0, 1) matrix ($n > m_0$).

A) If its rank is n, then any additional column vector is linearly dependent of the column vectors of the matrix.

B) 1. If its rank is $k < n$ and its row-degree is $l > \log n$ then by Lemma 2 we have at most $\frac{2 \cdot 2^n}{\sqrt{l}} < \frac{2 \cdot 2^n}{\sqrt{\log n}}$ possibilities to add a column vector so that the rank of the obtained $n \times (n+1)$ is also k.

B) 2. The number of those $n \times (n+1)$ (0, 1) matrices for which the row-degree of the $n \times n$ matrix consisting of its first n columns is less than $\log n$ but not equal to 1 — by Lemma 4 — is less than $2^{n(n+1)} \cdot \frac{2^{n^{4/5}}}{2^n}$.

B) 3. If an $n \times (n+1)$ matrix has the property that the row-degree of the $n \times n$ matrix consisting of its first n columns is equal to 1, then (because $k < n$) in the latter matrix there exist two rows which are equivalent. So the number of these matrices is less than $2^{n(n+1)} \cdot \frac{n^2}{2^n}$.

Let **B** denote the set of matrices of the types B)2. and B)3. The number of elements of **B** is less than

$$2^{n(n+1)} \left(\frac{2^{n^{4/5}}}{2^n} + \frac{n^2}{2^n} \right) < 2^{n(n+1)} \cdot \frac{1}{2^{n/2}}$$

if $n > n_0 \geqq m_0$ for some suitable natural number n_0.

By a similar way we can prove that if we enlarge the obtained $n \times (n+1)$ matrix by a row vector and if the matrix is not an element of the set **B**, then the probability of the event, that the rank of the new matrix is larger than the rank of the first matrix is at least

$$1 - \frac{2}{\sqrt{\log n}}.$$

III.

a) So we have proved the following

LEMMA 5. *Let us consider an arbitrary $n \times n$ (0, 1) matrix which is not element of the set* **B**. *Let us enlarge the matrix by a column vector (with components 0, 1) and let us add to the new matrix a row vector in all possible ways. So we obtain 2^{2n+1} $(n+1)\times(n+1)$ matrix.*

If the rank of the first matrix is $k<n$, then the rank of the new matrices are $k+2$ except for at most $\frac{2}{\sqrt{\log n}}\cdot 2^{2n+1}$ matrices, and if the rank of the first matrix is $k=n$, then the rank of the new matrices are $n+1$ except for at most $\frac{2}{\sqrt{\log n}}\cdot 2^{2n+1}$ matrices.

b) Using Lemma 5 we obtain

LEMMA 6. *There exists a sequence $n_1, n_2, \ldots, n_k, \ldots$ of natural numbers such that*

$$A_{n_k,n_k} > 2^{n_k^2}\left(1-\frac{6}{\sqrt{\log n_k}}\right) \qquad (k = 1, 2, \ldots),$$

where $A_{m,r}$ denotes the number of $m \times m$ (0, 1) matrices whose ranks are equal to r.

By other words

$$\liminf_{n=+\infty} p_n = \liminf_{n=+\infty} \frac{A_n}{2^{n^2}} = \limsup\left(1-\frac{A_{n,n}}{2^{n^2}}\right) = 1-1 = 0.$$

PROOF. Let us put $S_n = \sum_{k=0}^{n} A_{n,k}\cdot k$ and $f(n) = \frac{S_n}{2^{n^2}}$. The inequality $S_n <$ $< \sum_{k=0}^{n} A_{n,k}\cdot n = n\cdot 2^{n^2}$ implies that $f(n)<n$. Let $\bar{A}_{n,k}$ denote the number of those $n\times n$ matrices whose ranks are k and which are not elements of the set **B** and $B_{n,k} = A_{n,k} - \bar{A}_{n,k}$. We can obtain all $(n+1)\times(n+1)$ matrices so that we enlarge the $n \times n$ matrices by a column vector to the right and after it by a row vector upwards in all possible ways.

So we can obtain from the $n\times n$ matrices of number $\bar{A}_{n,k}$ and of rank k new $(n\times 1)+(n+1)$ matrices the number of which is $\bar{A}_{n,k}\,2^{2n+1}$ and among them $x_{n,k}\bar{A}_{n,k}2^{2n+1}$ have rank smaller than $\min(k+2, n+1)$. By Lemma 5.

$$x_{n,k} < \frac{2}{\sqrt{\log n}} \qquad (k = 0, 1, 2, \ldots, n).$$

c) So we have

$$S_{n+1}=\sum_{k=0}^{n+1}k\cdot A_{n+1,k}\geqq 2^{2n+1}\sum_{k=0}^{n-1}\bar{A}_{n,k}(1-x_{n,k})(k+2)+$$

$$+2^{2n+1}\sum_{k=0}^{n-1}\bar{A}_{n,k}\cdot x_{n,k}\cdot k+2^{2n+1}\bar{A}_{n,n}(1-x_{n,n})(n+1)+2^{2n+1}\bar{A}_{n,n}\cdot x_{n,n}\cdot n=$$

$$=2^{2n+1}\left(\sum_{k=0}^{n-1}\bar{A}_{n,k}(k+2)+\bar{A}_{n,n}(n+1)\right)-2^{2n+1}\left(2\cdot\sum_{k=0}^{n-1}\bar{A}_{n,k}\cdot x_{n,k}+\bar{A}_{n,n}x_{n,n}\right)\geqq$$

$$\geqq 2^{2n+1}\left(\sum_{k=0}^{n-1}\bar{A}_{n,k}(k+2)+\bar{A}_{n,n}(n+1)\right)-2^{2n+1}\left(2\cdot\sum_{k=0}^{n-1}\bar{A}_{n,k}+\bar{A}_{n,n}\right)\frac{2}{\sqrt{\log n}}=$$

$$=2^{2n+1}\left(\sum_{k=0}^{n}\bar{A}_{n,k}(k+2)-\bar{A}_{n,n}\right)-2^{2n+1}\left(2\cdot\sum_{k=0}^{n}\bar{A}_{n,k}-\bar{A}_{n,n}\right)\frac{2}{\sqrt{\log n}}=$$

$$=2^{2n+1}\sum_{k=0}^{n}\bar{A}_{n,k}\cdot k+2\cdot 2^{(n+1)^2}\left(1-\frac{2}{\sqrt{\log n}}\right)-2^{2n+1}\cdot\bar{A}_{n,n}\left(1-\frac{2}{\sqrt{\log n}}\right)-$$

$$-2\cdot 2^{2n+1}\left(1-\frac{2}{\sqrt{\log n}}\right)\sum_{k=0}^{n}B_{n,k}=2^{2n+1}\sum_{k=0}^{n}A_{n,k}\cdot k+2\cdot 2^{(n+1)^2}\left(1-\frac{2}{\sqrt{\log n}}\right)-$$

$$-2^{2n+1}\cdot A_{n,n}\left(1-\frac{2}{\sqrt{\log n}}\right)-2\cdot 2^{2n+1}\left(1-\frac{2}{\sqrt{\log n}}\right)\sum_{k=0}^{n}B_{n,k}-$$

$$-2^{2n+1}\sum_{k=0}^{n}B_{n,k}\cdot k+2^{2n+1}\cdot B_{n,n}\left(1-\frac{2}{\sqrt{\log n}}\right)\geqq$$

$$\geqq 2^{2n+1}\cdot S_n+2\cdot 2^{(n+1)^2}\left(1-\frac{2}{\sqrt{\log n}}\right)-2^{2n+1}\cdot A_{n,n}\left(1-\frac{2}{\sqrt{\log n}}\right)-$$

$$-2^{2n+1}\cdot 2^{n^2}\frac{n+2}{2^{n/2}}\geqq 2^{2n+1}\cdot S_n+2\cdot 2^{(n+1)^2}\left(1-\frac{3}{\sqrt{\log n}}\right)-2^{2n+1}\cdot A_{n,n}\left(1-\frac{2}{\sqrt{\log n}}\right).$$

d) Dividing by $2^{(n+1)^2}$ we get

$$f(n+1)\geqq f(n)+2\left(1-\frac{3}{\sqrt{\log n}}\right)-\frac{A_{n,n}}{2^{n^2}}\left(1-\frac{2}{\sqrt{\log n}}\right).$$

If we suppose that there exists a number N_0 such that

$$A_{n,n}<2^{n^2}\left(1-\frac{6}{\sqrt{\log n}}\right)$$

holds for all $n \geqq N_0$ then we have

$$f(n+1) \geqq f(n)+2\left(1-\frac{3}{\sqrt{\log n}}\right)-\left(1-\frac{7}{\sqrt{\log n}}\right)$$

that is

$$f(n+1) \geqq f(n)+\left(1+\frac{1}{\sqrt{\log n}}\right) \tag{1}$$

for all $n \geqq N_0$. But $\sum_{k=N_0}^{\infty} \frac{1}{\sqrt{\log k}} = +\infty$ therefore using Relation (1) $(n-N_0)$ times we obtain

$$f(n+1) \geqq f(N_0)+(n-N_0)+\sum_{k=N_0}^{n} \frac{1}{\sqrt{\log k}}.$$

for all $n \geqq N_0$. If N is so large that $\sum_{k=N_0}^{N} \frac{1}{\sqrt{\log k}} > N_0+1$ holds, then we have $f(N+1) > N+1$ which is a contradiction. Q.e.d.

IV.

a) LEMMA 7.

Let $f(x, y)$ be a function defined for all pairs $x \geqq y$ of natural numbers with the following properties:

There exists a natural number n and a real number $0<c<1$ such that

1° $f(x, y) \geqq 0$
2° $f(x, x)=1$
3° $f(x, y+1) \geqq f(x, y)$
4° $f(n, n-1)<c$
5° $f(m+1, k) \leqq cf(m, k)+(1-c)f(m, k-2)+d_m$

for all $m \geqq n$ and $0 \leqq k \leqq m$, where $\{d_m\}$ is a sequence of positive numbers.

We show that these properties imply that

$$f(m, m-1) < 2c+\sum_{s=n}^{\infty} d_s \tag{2}$$

or all $m \geqq n$.

b) By a double application of 5° we get

$$f(m+2, k) \leqq c^2 f(m, k)+2c(1-c)f(m, k-2)+(1-c)^2 f(m, k-4)+d_m+d_{m+1} \tag{3}$$

$$\begin{pmatrix} m \geqq n \\ 0 \leqq k \leqq m \end{pmatrix}$$

and this inequality implies $\big($as $f(m, k-4) \leqq f(m, k-2)\big)$:

$$(4)\qquad f(m+2,k)\leqq c^2 f(m,k)+(1-c^2)f(m,k-2)+d_m+d_{m+1}.$$
$$\begin{pmatrix} m\geqq n \\ 0\leqq k\leqq m\end{pmatrix}$$

The relation

$$f(N+1,k)\leqq cf(N,k)+(1-c)f(N,k-2)+d_N\leqq$$
$$\leqq f(N,k)[c+(1-c)]+d_N=f(N,k)+d_N$$
$$\begin{pmatrix} N\geqq n \\ 0\leqq k\leqq N\end{pmatrix}$$

and Relation 1° $(f(n,n-1)<c)$ show that

$$(5)\qquad f(m,k)<c+\sum_{s=n}^{m-1} d_s \quad \text{for all} \quad k\leqq n-1 \qquad (m\geqq n).$$

Now we prove by induction that the following inequality holds:

$$(6)\qquad f(n+t,n-2+t-i)\leqq c+\sum_{s=0}^{\left[\frac{t-i}{2}\right]-1}\binom{i+s}{s}c^{i+s+2}+\sum_{s=n}^{n+t-1} d_s$$
$$(t\geqq 2,\ i\geqq 0).$$

If $i>t-2$, then we have to prove that $f(n+t,n-(i-t+2))\leqq c+\sum_{s=n}^{n+t-1} d_s$; but this is an immediate consequence of (5). Let us suppose that

$$i\leqq t-2.$$

c) In the case $t=2$ (and so $i=0$) the inequality is

$$f(n+2,n)\leqq c+c^2+d_n+d_{n+1}.$$

By (4) we have

$$f(n+2,n)\leqq c^2 f(n,n)+(1-c^2)f(n,n-2)+d_n+d_{n+1}\leqq$$
$$\leqq c^2+(1-c^2)c+d_n+d_{n+1}<c+c^2+d_n+d_{n+1}.$$

In the case $t=3$ the inequality is (for $i=1$ or $i=0$)

$$f(n+3,n)\leqq c+c^3+d_n+d_{n+1}+d_{n+2}$$
$$f(n+3,n+1)\leqq c+c^2+d_n+d_{n+1}+d_{n+2}.$$

Using Relation (4):

$$f(n+3,n)\leqq c^2 f(n+1,n)+(1-c^2)f(n+1,n-2)+d_{n+1}+d_{n+2}\leqq$$
$$\leqq c^2[cf(n,n)+(1-c)f(n,n-2)+d_n]+(1-c^2)(c+d_n)+d_{n+1}+d_{n+2}\leqq$$
$$\leqq c^3+c^3(1-c)+c(1-c^2)+d_n+d_{n+1}+d_{n+2}<$$
$$<c+c^3+d_n+d_{n+1}+d_{n+2}$$

or similarly:

$$f(n+3, n+1) \leqq c^2 f(n+1, n+1) + (1-c^2) f(n+1, n-1) + d_{n+1} + d_{n+2} \leqq$$
$$\leqq c^2 + (1-c^2)(c+d_n) + d_{n+1} + d_{n+2} <$$
$$< c + c^2 + d_n + d_{n+1} + d_{n+2}.$$

That is the inequality is proved in the cases $t=2$ and $t=3$.

d) Let us suppose that the inequality is proved for $t=T$ and let us prove it for $t=T+2$. Denote $\left[\frac{T-i}{2}\right] = w$. Applying (3) we get, if $i \geqq 2$ $\left(\binom{n}{k} = 0 \text{ per. def. if } k > n \text{ or } k < 0\right)$

$$f(n+T+2, n-2+(T+2)-i) = f(n+T+2, n+T-i) \leqq c^2 f(n+T, n+T-i) +$$
$$+ 2c(1-c) f(n+T, n-2+T-i) + (1-c)^2 f(n+T, n-4+T-i) + d_{n+T} + d_{n+T+1} \leqq$$
$$\leqq c^2 \left(c + \sum_{s=0}^{w} \binom{i+s-2}{s} c^{i+s} + \sum_{s=n}^{n+T-1} d_s\right) +$$
$$+ 2c(1-c) \left(c + \sum_{s=0}^{w-1} \binom{i+s}{s} c^{i+s+2} + \sum_{s=n}^{n+T-1} d_s\right) +$$
$$+ (1-c)^2 \left(c + \sum_{s=0}^{w-2} \binom{i+s+2}{s} c^{i+s+4} + \sum_{s=n}^{n+T-1} d_s\right) + d_{n+T} + d_{n+T+1} =$$
$$= c + \sum_{s=n}^{n+T+1} d_s + \sum_{s=0}^{w} \binom{i+s-2}{s} c^{i+s+2} + 2 \sum_{s=1}^{w} \binom{i+s-1}{s-1} c^{i+s+2} -$$
$$- 2 \sum_{s=2}^{w+1} \binom{i+s-2}{s-2} c^{i+s+2} + \sum_{s=2}^{w} \binom{i+s}{s-2} c^{i+s+2} - 2 \sum_{s=3}^{w+1} \binom{i+s-1}{s-3} c^{i+s+2} +$$
$$+ \sum_{s=4}^{w+2} \binom{i+s-2}{s-4} c^{i+s+2} = S.$$

Using the following identity

$$\binom{i+s-2}{s} + 2\binom{i+s-1}{s-1} - 2\binom{i+s-2}{s-2} + \binom{i+s}{s-2} - 2\binom{i+s-1}{s-3} + \binom{i+s-2}{s-4} = \binom{i+s}{s}$$

(this identity holds for $s \geqq 1$, $i \geqq 1$)

one can see, that

$$S = c + \sum_{s=n}^{n+T+1} d_s + \sum_{s=0}^{w} \binom{i+s}{s} c^{i+s+2} - 2\binom{i+w-1}{w-1} c^{i+w+3} -$$
$$- 2\binom{i+w}{w-2} c^{i+w+3} + \binom{i+w}{w-2} c^{i+w+4} + \binom{i+w-1}{w-3} c^{i+w+3},$$

and as

$$\binom{i+w}{w-2}+\binom{i+w-1}{w-3} \leqq 2\binom{i+w}{w-2},$$

we get the relation

$$f(n+(T+2), n-2+(T+2)-i) \leqq S \leqq c+\sum_{s=0}^{\left[\frac{T-i}{2}\right]}\binom{i+s}{s} c^{i+s+2}+\sum_{s=n}^{n+T+1} d_s$$

what we had to prove.

If $i=0$ or $i=1$, then the estimate

$$f(n+T, n+T-i) \leqq c+\sum_{s=0}^{w}\binom{i+s-2}{s} c^{i+s}+\sum_{s=n}^{n+T-1} d_s$$

and also the identity was false. Instead of this estimate we write $f(n+T, n+T-i) \leqq 1$, and so we get for S the same formula as above.

e) Let us apply the proved inequality in the case $i=0$.

$$f(n+t, n+t-2) \leqq c+\sum_{s=0}^{\left[\frac{t}{2}\right]-1} c^{s+2}+\sum_{s=n}^{n+t-1} d_s \qquad (t \geqq 2).$$

Hence

$$f(n+t+1, n+t) \leqq c f(n+t, n+t)+(1-c) f(n+t, n+t-2)+d_{n+t} \leqq$$

$$\leqq c+(1-c)\left(c+\sum_{s=0}^{\left[\frac{t}{2}\right]-1} c^{s+2}+\sum_{s=n}^{n+t-1} d_s\right)+d_{n+t}<c+(1-c)\left(c+\sum_{s=0}^{\infty} c^{s+2}\right)+$$

$$+\sum_{s=n}^{\infty} d_s=c+c-c^2+(1-c) \frac{c^2}{1-c}+\sum_{s=0}^{\infty} d_s=2 c+\sum_{s=n}^{\infty} d_s$$

for all $t \geqq 2$.

But

$$f(n+2, n+1) \leqq c f(n+1, n+1)+(1-c) f(n+1, n-1)+d_{n+1} \leqq$$

$$\leqq c+(1-c)(c+d_n)+d_{n+1}<2 c+\sum_{s=n}^{\infty} d_s$$

and

$$f(n+1, n) \leqq c f(n, n)+(1-c) f(n, n-2)+d_n \leqq c+(1-c) c+d_n<2 c+\sum_{s=n}^{\infty} d_s,$$

hence we proved that for all $m \geqq n$

$$f(m, m-1)<2 c+\sum_{s=n}^{\infty} d_s$$

holds. Q. e.d.

V.

Now we can already prove the theorem:

Let ε be an arbitrary positive number. Let the integer N be so large that for the N-th element of the sequence n_k (defined in Lemma 6)

$$\frac{13}{\sqrt{\log n_N}} < \varepsilon.$$

Let us put

$$f(m, k) = \sum_{i=0}^{k} \frac{A_{m,i}}{2^{m^2}},$$

$$c = \frac{6}{\sqrt{\log n_N}},$$

$$n = n_N,$$

$$d_m = \frac{1}{2^{m/2}}.$$

It is easy to see that for the function $f(m, k)$ 1°—2°—3° hold.

The fulfilment of 4° follows from the definition of the sequence $\{n_k\}$ (in lemma 6). Let us prove that 5° holds.

From the $\bar{A}_{m,k-1}$ matrices of rank $k-1$ except for at most $c\cdot\bar{A}_{m,k-1}\cdot 2^{2m+1}$ ones, and from the $\bar{A}_{m,k}$ matrices of rank k except for at most $c\cdot\bar{A}_{m,k}\cdot 2^{2m+1}$ ones we get such matrices, which have at least $k+1$ as rank. So we have

$$2^{(m+1)^2} f(m+1, k) \leqq \sum_{i=0}^{k-2} A_{m,i}\cdot 2^{2m+1} + c\cdot 2^{2m+1}(\bar{A}_{m,k-1}+\bar{A}_{m,k}) +$$

$$+ 2^{2m+1}\cdot d_m\cdot 2^{m^2} \leqq \sum_{i=0}^{k-2} A_{m,i}\cdot 2^{2m+1} + c\cdot 2^{2m+1}(A_{m,k-1}+A_{m,k}) + d_m\cdot 2^{(m+1)^2} =$$

$$= f(m, k-2)\cdot 2^{(m+1)^2} + c\cdot 2^{(m+1)^2}\big(f(m,k)-f(m,k-2)\big) + d_m\cdot 2^{(m+1)^2} =$$

$$= 2^{(m+1)^2}[cf(m,k)+(1-c)f(m,k-2)+d_m].$$

Dividing by $2^{(m+1)^2}$ we obtain

$$f(m+1, k) \leqq cf(m,k)+(1-c)f(m,k-2)+d_m,$$

that is 5° holds.

By lemma 7 we get:

$$f(m, m-1) < 2c + \sum_{s=n}^{\infty} d_s$$

for all $m \geqq n$. But

$$\sum_{s=n}^{\infty} d_s = \sum_{s=n}^{\infty} \frac{1}{2^{s/2}} = \frac{4}{2^{n/2}} < \frac{1}{\sqrt{\log n}}$$

that is

$$f(m, m-1) < \frac{12}{\sqrt{\log n_N}} + \frac{1}{\sqrt{\log n_N}} < \varepsilon$$

for all $m \geqq n_N$ or in other terms

$$A_{m,m} > 2^{m^2}(1-\varepsilon) \quad \text{for all} \quad m \geqq n_N,$$

what proves our theorem.

VI.

a)

Professor EGYED asked whether the following generalization of this theorem is true:

Let us consider the matrices:

$$\begin{matrix} a_{1,1} & a_{1,2} \cdots a_{1,k} \cdots \\ a_{2,1} & a_{2,2} \cdots a_{2,k} \cdots \\ \cdot & \cdot \quad \cdot \\ a_{i,1} & a_{i,2} \cdots a_{i,k} \cdots \\ \vdots & \vdots \qquad \vdots \end{matrix}$$

where the elements $a_{i,k}$ equal to 0 or 1. The set of those matrices in which the rows or the columns are not "linearly independent", has a measure 0.

First we have to agree in that what is the meaning of "linearly independent" in this case.

Let $a_{i,k}$ $(i=1, 2, \ldots; k=1, 2, \ldots)$ be mutually independent random variables which take on the values 0, 1 with probabilities $\frac{1}{2}, \frac{1}{2}$. Let us form by these random variables the above matrix.

We make use of two definitions of the linear dependence of the rows of a matrix.

The rows of a matrix are *finitely linearly dependent*, if there exists a natural number i, some natural numbers (finitely many) $i_1 < i_2 < \ldots < i_s$ and real numbers $\alpha_1, \alpha_2, \ldots, \alpha_s$ with the properties:

$$i_\nu \neq i \quad \text{for} \quad \nu = 1, 2, \ldots, s$$

and

$$a_{i,k} = \sum_{\nu=1}^{s} \alpha_\nu a_{i_\nu,k} \quad \text{for} \quad k = 1, 2, \ldots. \tag{7}$$

The rows of a matrix are *infinitely linearly dependent*, if there exists a natural number i and real numbers $\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_i = 0, \alpha_{i+1}, \ldots$ such that

$$a_{i,k} = \sum_{\nu=1}^{\infty} \alpha_\nu a_{\nu,k} \quad \text{for} \quad k = 1, 2, \ldots. \tag{8}$$

Let A denote the event that the rows of a random matrix are finitely linearly dependent and B the event that they are infinitely linearly dependent.

Making use of these definitions we can formulate the question as follows: What are the probabilities $\mathbf{P}(A)$ and $\mathbf{P}(B)$ equal to?

b) *The answer is:*

$$\mathbf{P}(A)=0, \tag{9}$$

$$\mathbf{P}(B)=1. \tag{10}$$

The proofs of these relations are simple.

Proof of (9):

Let A_t denote the event that

$$a_{i_1,t}=a_{i_2,t}=\ldots=a_{i_s,t}=0 \quad \text{and} \quad a_{i,t}=1.$$

Clearly $A_1, A_2, \ldots, A_t, \ldots$ are mutually independent and $0<\mathbf{P}(A_1)=\mathbf{P}(A_2)=\ldots=$ $=\mathbf{P}(A_t)=\ldots$. One can see from the relation (7) that $A_t\cap A=\emptyset$, so $\left(\bigcup_{t=1}^{\infty} A_t\right)\cap A=\emptyset$ and thus $\overline{\bigcup_{t=1}^{\infty} A_t}=\bigcap_{t=1}^{\infty}\bar{A}_t\supset A$. This fact implies that

$$\mathbf{P}\left(\bigcap_{t=1}^{\infty}\bar{A}_t\right)\geqq\mathbf{P}(A).$$

But we have $\mathbf{P}\left(\bigcap_{t=1}^{\infty}\bar{A}_t\right)=\prod_{t=1}^{\infty}\mathbf{P}(\bar{A}_t)$ because the events $\bar{A}_t$ are independent.

As $\mathbf{P}(\bar{A}_t)<1$ and

$$\mathbf{P}(\bar{A}_1)=\mathbf{P}(\bar{A}_2)=\ldots=\mathbf{P}(\bar{A}_t)=\ldots,$$

we get $\prod_{t=1}^{\infty}\mathbf{P}(\bar{A}_t)=0$ whence $\mathbf{P}(A)=0$.

Proof of (10):

Let B_t denote the event $(t=1, 2, \ldots)$ that there exists a natural number i_t (different from $i_1, i_2, \ldots, i_{t-1}$) such that

$$a_{i_t,1}=a_{i_t,2}=\ldots=a_{i_t,t-1}=0 \quad \text{and} \quad a_{i_t,t}=1.$$

Clearly $\mathbf{P}(B_t)=1$, therefore $\mathbf{P}\left(\bigcap_{t=1}^{\infty}B_t\right)=1$.

That is, a random matrix contains a triangular matrix, in which all diagonal elements are equal to 1, with probability 1. Clearly the matrix also contains an i-th row vector which is different from the rows of the triangular matrix, with probability 1. We show that such a row of the matrix is an infinite linear combination of the i_1-th, i_2-th, ..., i_t-th, ... rows.

Put $\alpha_{i_1}=a_{i,1}$ and define the numbers α_{i_k} successively as

$$\alpha_{i_k}=a_{i,k}-\sum_{\nu=1}^{k-1}\alpha_{i_\nu}a_{i_\nu,k}.$$

If t is not one of the numbers i_k then let $\alpha_t=0$.

Since

$$a_{i_k,k}=1,$$

$$a_{i_\nu,k}=0 \quad \text{for} \quad \nu>k$$

and

$$\alpha_\nu=0 \quad \text{if} \quad \nu\neq i_t,$$

we have

$$a_{i,k}=\alpha_{i_k}+\sum_{\nu=1}^{k-1}\alpha_{i_\nu}a_{i_\nu,k}=\sum_{\nu=1}^{k}\alpha_{i_\nu}a_{i_\nu,k}=\sum_{\nu=1}^{\infty}\alpha_{i_\nu}a_{i_\nu,k}=\sum_{\mu=1}^{\infty}\alpha_\mu a_{\mu,k},$$

that is (10) holds.

The condition (8) says that

$$\lim_{N=+\infty}\sum_{\nu=1}^{N}\alpha_\nu a_{\nu,k}=a_{i,k} \quad \text{for} \quad k=1,2,\ldots.$$

If we substitute this condition by the condition

$$\lim_{N=+\infty}\sum_{\nu=1}^{N}\alpha_\nu a_{\nu,k}=a_{i,k} \text{ uniformly in } k,$$

then the probability in question is equal to 0.

c)

In the proof of (10) we actually proved that the rows (and clearly the columns too) of a random matrix contain an infinite basis with probability 1.

(A subset of a set of vectors is called) "basis infinitely", if any element of the set can uniquely be represented by an infinite linear combination of the elements of the subset.

A subset of a set of vectors is called to be "basis finitely" (it can contain infinitely many vectors) if any element of the set can uniquely be represented by a linear combination of finitely many vectors of the elements of the subset).

I do not know whether there exists a set of vectors (with countably many components) containing no "basises infinitely".

(Finitely many vectors ever contain basis. It is easy to see that a set of countably many vectors also contain at least one "basis finitely" and we proved above that this basis is the whole set with probability 1.)

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