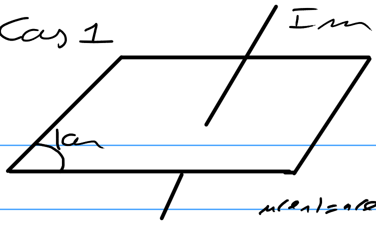
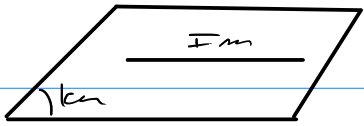
Quelques endomorphismes de rang 1

Cas 1 

Cas 2 

$u(e_n) = n(e_1)$

(1) $$A = \begin{array}{c} \overbrace{u(e_1) \; \cdots \; u(e_n)} \\ \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & (1) & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \end{array} \begin{array}{c} e_1 \\ \vdots \\ e_n \end{array} \in \mathcal{M}_n(\mathbb{R})$$ (S/2) symétrique réelle

$\mathrm{rg}(A) = \mathrm{rg}\, u = 1$ $\quad \underbrace{\mathrm{Ker}(A)}_{\mathrm{Ker}(A - 0 I_n)}$ dim $\underline{n-1}$

$$\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} n \\ \vdots \\ n \end{pmatrix}$$

$\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ vep pour la vap n

$$\underbrace{\mathrm{Ker}(A)}_{n-1} \oplus \underbrace{\mathrm{Ker}(A - n I_n)}_{\geq 1} \overset{=}{\subset} \underbrace{E}_{n}$$

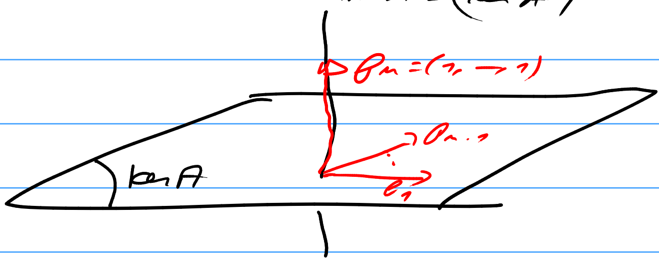
$$D = \begin{array}{c} u(e_1) \; \cdots \; u(e_{n-1}) \; u(e_n) \\ \begin{pmatrix} 0 & & & 0 \\ & \ddots (0) & & \vdots \\ & & 0 & 0 \\ 0 & \cdots & 0 & n \end{pmatrix} \end{array} \begin{array}{c} e_1 \\ \vdots \\ e_n \end{array}$$

$u(e_n) = n e_n$

$$P = \begin{array}{c} e_1 \; \cdots \; e_{n-1} \; e_n \\ \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & 0 & \cdots & 0 & \vdots \\ 0 & -1 & & \vdots & \vdots \\ \vdots & 0 & & 0 & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \end{array} \begin{array}{c} e_1 \\ \vdots \\ e_n \end{array}$$

$$P^{-1} \underbrace{A}_{\mathrm{tr} = n} P = \underbrace{D}_{\mathrm{tr} = n}$$

$\mathrm{Im}\, A = (\mathrm{Ker}\, A)^{\perp}$



Cousines de A ! $\quad \bullet \; B = \begin{pmatrix} 0 & & (1) \\ & \ddots & \\ (1) & & 0 \end{pmatrix} = A - I_n$

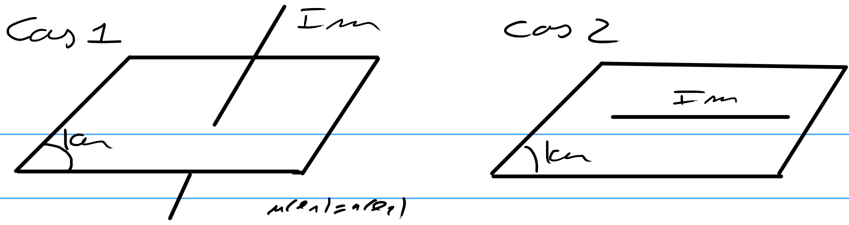
$A = PDP^{-1} \qquad I = PIP^{-1}$

$$B = PDP^{-1} - PIP^{-1} = P \underbrace{(D - I)}_{\begin{pmatrix} -1 & & (0) \\ & \ddots \, -1 & \\ (0) & & n-1 \end{pmatrix}} P^{-1}$$

Si V vep de A

Alors : $BV = (A - I)V = AV - V = \lambda V - V = \underline{(\lambda - 1)} V$

$$\bullet \begin{pmatrix} a & & (b) \\ & \ddots & \\ (b) & & a \end{pmatrix} = \begin{pmatrix} (b) \end{pmatrix} + \begin{pmatrix} a-b & & (0) \\ & \ddots & \\ (0) & & a-b \end{pmatrix}$$
$$= bA + (a-b) I_n$$



② $\Phi \left\| \begin{array}{l} M \longmapsto tr(M)\, A^{\neq 0} \\ \underbrace{\mathcal{M}_n(\mathbb{K})}_{E \to \dim n^2} \longrightarrow \mathcal{M}_n(\mathbb{K}) \end{array}\right.$ $\quad A \in \mathcal{M}_n(\mathbb{K})$

• $\ker \Phi = \{M \in E;\ \Phi(M)=0\} = \{M;\ tr\, M = 0\} = \ker(tr)$

$tr : \mathcal{M}_n(\mathbb{K}) \longrightarrow \mathbb{K}$: "forme linéaire"

$\ker \Phi = \ker(tr)$: hyperplan : $\dim = n^2-1$

• $Im\, \Phi = \{\underbrace{\Phi(M)}_{tr(M)A},\ M \in \mathcal{M}_n(\mathbb{K})\} \subseteq Vect(A)$

$rg\, \Phi = 1$

→ $\ker\Phi \cap \overline{Im\,\Phi} \overset{?}{=} \{0\}$

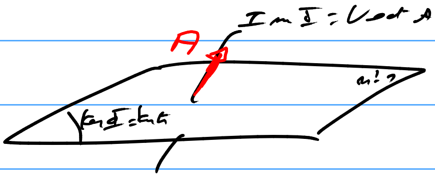
→ $\underline{Im\,\Phi \overset{?}{\subset} \ker\Phi}$

$A \notin \ker\Phi$

$\underset{tr(A)A}{\Phi(A)} \neq 0$

$\boxed{\text{Si } tr\, A \neq 0}$ $\quad A \in \ker\Phi$ donc $Vect\, A \cap \ker\Phi = \{0\}$

$\ker\Phi \oplus Im\,\Phi = E$



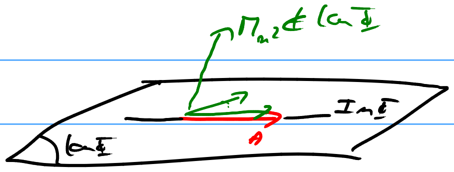
$Mat(\Phi, \mathfrak{B}) =$

''

$$\begin{array}{c} \Phi(\ ) \cdots \Phi(\ )\ \Phi(A) \\ \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & tr(A) \end{pmatrix} \end{array} \begin{array}{l} \} n^2-1 \\ \\ A \end{array}$$

Φ est DZ

$\boxed{\text{Si } tr\, A = 0}$ $\quad A \in \ker\Phi$ donc $\underbrace{Vect\, A}_{Im\,\Phi} \subset \ker\Phi$



$$Mat\, \Phi = \begin{array}{c} \Phi(\ ) \cdots \Phi(\ )\ \Phi(M_{n^2}) \\ \begin{pmatrix} 0 & & 0 & \alpha \\ & & & 0 \\ \vdots & 0 & & \vdots \\ 0 & & 0 & 0 \end{pmatrix} \end{array} \begin{array}{l} A \\ \vdots \\ \\ M_{n^2} \end{array}$$

$Mat(\Phi, \mathcal{C}) = \begin{pmatrix} 0 & & (0) \\ & \ddots & \\ (0) & & 0 \end{pmatrix} = 0$

$\Phi = 0$: EXCLU $\quad \Phi$ n'est PAS DZ

$$\Psi : M \longmapsto \underbrace{\mathrm{tr}(M)A}_{\Phi(M)} + \underbrace{\mathrm{tr}(A)M}_{k\,M}$$

$$\Psi = \Phi + k\,\mathrm{Id}.$$

• Si $\mathrm{tr}\,A \neq 0$: Ψ DZ

$$\begin{pmatrix} \mathrm{tr}(A) & & (0) \\ & \ddots & \\ (0) & (\mathrm{tr}\,A) & 2\,\mathrm{tr}\,A \end{pmatrix}$$

• Si $\mathrm{tr}\,A = 0$:

$$\begin{pmatrix} \mathrm{tr}\,A & & \alpha \\ & \ddots & \\ (0) & & \mathrm{tr}\,A \end{pmatrix}$$

Ψ non DZ

Si Ψ l'était : $\mathrm{Sp}\,\Psi = \{\mathrm{tr}\,A\}$

$$E = \mathrm{Ker}(\Psi - \mathrm{tr}\,A\,\mathrm{Id})$$

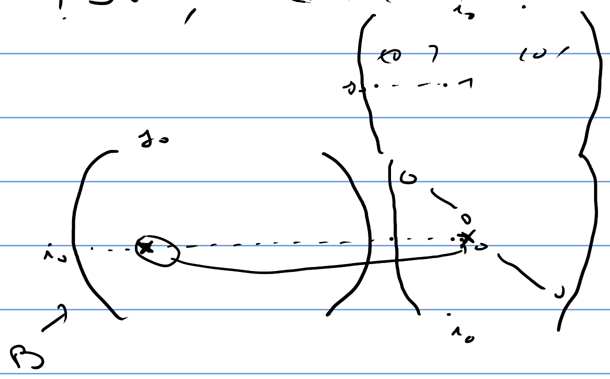
$$\Psi = \mathrm{tr}(A)\,\mathrm{Id}$$

$$\forall M,\ \Psi(M) = \mathrm{tr}(A)M \quad \text{FAUX}$$

$$F \left| \begin{array}{l} M \longmapsto \mathrm{tr}(BM)\,I_n \\ \mathcal{M}_n(\mathbb{K}) \longrightarrow \mathcal{M}_n(\mathbb{K}) \end{array} \right. \qquad B \neq 0$$

$$\mathrm{Im}(F) \subset \mathrm{Vect}(I_n)$$

$$? \exists M ;\ \mathrm{tr}(BM) \neq 0 \ ??$$



$\mathrm{Im}(F) = \mathrm{Vect}(I_n)$

$\cap$

$\mathrm{Vect}\,I_n$

$F(I_n) = 0$??

$\mathrm{tr}\,B = 0$ ou pas !

③ $M = C.L$

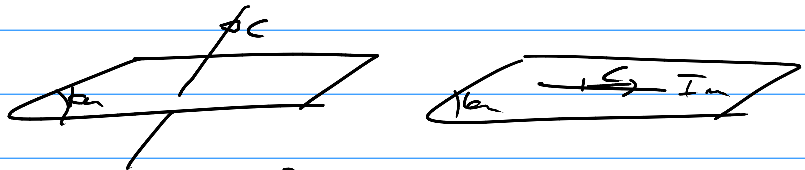
$$\begin{pmatrix} 0 & 1 & -2 \\ 0 & 2 & -4 \\ 0 & -3 & 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \begin{pmatrix} 0 & 1 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & -2 \\ 0 & 2 & -4 \\ 0 & \underbrace{-3}_{C_2} & \underbrace{6}_{-2C_2} \end{pmatrix}$$

$rg(M) = 1 \iff \exists C \neq 0, L \neq 0 : \underline{M = CL}$ $\quad C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$, $L = (l_1 \dots l_n)$

$$M^2 = (CL)(CL) = C(\underbrace{LC}_{})L = (\operatorname{tr} M) M$$

$$\sum l_i c_i = \operatorname{tr} M$$



$C \notin \operatorname{Ker} M$

$$MC = (CL)C = C(\underbrace{LC}_{\operatorname{tr} M}) = \underbrace{(\operatorname{tr} M)}_{}C$$

$u(\vec{x}) = \lambda \vec{x}$

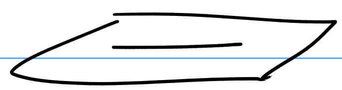
- Si $\operatorname{tr} M \neq 0$: on a une nelle vap

$$\underbrace{\operatorname{Ker}(M)}_{n-1} \oplus \underbrace{\operatorname{Ker}(M - \operatorname{tr}(M) I)}_{\geq 1} \underset{=}{\subset} E$$

D2

$$M^2 - (\operatorname{tr} M) M = 0 : \underbrace{X^2 - \operatorname{tr}(M) X}_{X(X - \underset{\neq 0}{\operatorname{tr}(M)})}$$

- Si $\operatorname{tr} M = 0$



$\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$

M non D2

$M^2 = 0) \quad Sp(M) \subset \underbrace{Rac(X^2)}_{\{0\}}$

$Sp(M) = \{0\}$ $\quad P \begin{pmatrix} 0 & \\ & 0 \end{pmatrix} P^{-1}$

$$M = P \begin{pmatrix} 0 & \\ & 0 \end{pmatrix} P^{-1} = 0$$

- Vari G $\quad I + CL$

$$\underbrace{\underbrace{I+CL}_{PDP^{-1}}}_{PP^{-1}+PDP^{-1} = P(I+D)P^{-1}}$$

- $$\begin{pmatrix} x_1^2 & \cdots & x_1 x_n \\ x_1 x_2 & & \vdots \\ \vdots & x_i x_j & \vdots \\ x_n x_1 & & x_n x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} (x_1 \cdots x_n)$$

$$C\,{}^tC \qquad G = \underbrace{\sum_{i=1}^{n} x_i^2}_{} = 0\ ??$$

$$\begin{pmatrix} 1 \\ i \end{pmatrix}(1\ \ i) \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \leftarrow \text{pas DZ !}$$

$$(x-1)(x+1) - i^2 = x^2 \quad) \ S_p = \{0\}$$

- $$\begin{pmatrix} 1 & a & & a^{n-1} \\ a & a^2 & & \vdots \\ a^2 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ a^{n-1} & a^n & & a^{2n-2} \end{pmatrix} \quad a^{i+j-2}$$

$$\begin{pmatrix} 1 \\ a \\ \vdots \\ a^{n-1} \end{pmatrix} (1\ a \cdots a^{n-1}) \qquad G = 0 ?$$

$$1 + a^2 + \cdots + (a^2)^{n-1} \neq 0$$

$$\begin{pmatrix} 1 \\ i \end{pmatrix}(1\ \ i) \qquad \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$$