Régulateur Diravi : Corrigé

① Le rotor 1 ayant 2 plans de symetrie $(O, \vec{x_1}, \vec{y_1})$ et $(O, \vec{y_1}, \vec{z_1})$

$$\overline{\overline{J_O(1)}} = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & C_1 \end{pmatrix}_{R_1}$$

Les masselottes ayant 1 plan de symetrie $(A, \vec{x_2}, \vec{y_2})$

$$\overline{\overline{J_A(2)}} = \begin{pmatrix} A_2 & -F_2 & 0 \\ -F_2 & B_2 & 0 \\ 0 & 0 & C_2 \end{pmatrix}_{R_2}$$

La bague 3 etant un solide de revolution d'axe $(O\vec{y_1}) = (G_3\vec{y_1})$

$$\overline{\overline{J_{G_3}(3)}} = \begin{pmatrix} A_3 & 0 & 0 \\ 0 & B_3 & 0 \\ 0 & 0 & A_3 \end{pmatrix}_{R_1}$$

② $\overrightarrow{\Omega_{1/0}} = \dot{\alpha}\, \vec{y_1}$

$\overrightarrow{V_{A\in 1/0}} = \overrightarrow{V_{O\in 1/0}} + \overrightarrow{AO} \wedge \overrightarrow{\Omega_{1/0}} = \vec{0} + r\, \vec{x_1} \wedge \dot{\alpha}\vec{y_1} = r\dot{\alpha}\, \vec{z_1}$

$\overrightarrow{\Omega_{2/0}} = \overrightarrow{\Omega_{2/1}} + \overrightarrow{\Omega_{1/0}} = \dot{\beta}\, \vec{z_2} + \dot{\alpha}\, \vec{y_1}$

$$\overrightarrow{\Omega_{2/0}} = \begin{pmatrix} \dot{\alpha} \sin\beta \\ \dot{\alpha}\cos\beta \\ \dot{\beta} \end{pmatrix}_{R_2}$$

$\overrightarrow{V_{A\in 2/0}} = \overrightarrow{V_{A\in 1/0}} = r\dot{\alpha}\, \vec{z_1} = r\dot{\alpha}\, \vec{z_2}$

$$\overrightarrow{V_{G\in 2/0}} = \overrightarrow{V_{A\in 2/0}} + \overrightarrow{GA} \wedge \overrightarrow{\Omega_{2/0}} = \begin{pmatrix} 0 \\ 0 \\ r\dot{\alpha} \end{pmatrix}_{R_2} + \begin{pmatrix} d \\ -\ell \\ 0 \end{pmatrix}_{2} \wedge \begin{pmatrix} \dot{\alpha}\sin\beta \\ \dot{\alpha}\cos\beta \\ \dot{\beta} \end{pmatrix}_{R_2}$$

$$\overrightarrow{V_{G\in 2/0}} = \begin{pmatrix} -\ell\dot{\beta} \\ -d\dot{\beta} \\ (r + d\cos\beta + \ell\sin\beta)\dot{\alpha} \end{pmatrix}_{R_2}$$

③ 1ère méthode

→ On determine le moment cinetique en A :

$$\overrightarrow{\sigma_A(2/0)} = m_2\, \overrightarrow{AG} \wedge \overrightarrow{V_{A\in 2/0}} + \overline{\overline{J_A(2)}} \cdot \overrightarrow{\Omega_{2/0}}$$

→ Puis on determine le moment cinetique en G par la relation de Varignon :

$$\overrightarrow{\sigma_G(2/0)} = \overrightarrow{\sigma_A(2/0)} + \overrightarrow{GA} \wedge m_2 \overrightarrow{V_{G\in 2/0}}$$

2ᵉ méthode

→ On détermine par le théorème de Huygens la matrice d'inertie de 2 au point G. $\overline{\overline{J_G(2)}}$

→ On détermine ensuite le moment cinétique en G

$$\overrightarrow{\sigma_G(2/0)} = \overline{\overline{J_G(2)}} \cdot \overrightarrow{\Omega(2/0)}$$

1ʳᵉ méthode

$$\overline{\overline{J_A(2)}} \cdot \overrightarrow{\Omega_{2/0}} = \begin{pmatrix} A_2 & -F_2 & 0 \\ -F_2 & B_2 & 0 \\ 0 & 0 & C_2 \end{pmatrix} \cdot \begin{pmatrix} \dot\alpha \sin\beta \\ \dot\alpha \cos\beta \\ \dot\beta \end{pmatrix} = \begin{pmatrix} (A_2 \sin\beta - F_2 \cos\beta)\dot\alpha \\ (B_2 \cos\beta - F_2 \sin\beta)\dot\alpha \\ C_2 \dot\beta \end{pmatrix}_{R_2}$$

$$m_2 \overrightarrow{AG} \wedge \overrightarrow{V_{A\in 2/0}} = m_2 \begin{pmatrix} -d \\ \ell \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ r\dot\alpha \end{pmatrix} = \begin{pmatrix} m_2 . r \ell \dot\alpha \\ m_2 . r d \dot\alpha \\ 0 \end{pmatrix}_{R_2}$$

$$\overrightarrow{GA} \wedge m_2 \overrightarrow{V_{G\in 2/0}} = m_2 \begin{pmatrix} d \\ -\ell \\ 0 \end{pmatrix} \wedge \begin{pmatrix} -\ell\dot\beta \\ -d\dot\beta \\ (r + d\cos\beta + \ell \sin\beta)\dot\alpha \end{pmatrix}_{R_2}$$

$$= \begin{pmatrix} -m_2 \ell (r + d\cos\beta + \ell\sin\beta)\dot\alpha \\ -m_2 d (r + d\cos\beta + \ell\sin\beta)\dot\alpha \\ -m_2 (d^2 + \ell^2)\dot\beta \end{pmatrix}_{R_2}$$

$$\overrightarrow{\sigma_G(2/0)} = m_2 \overrightarrow{AG} \wedge \overrightarrow{V_{A\in 2/0}} + \overline{\overline{J_A(2)}} \cdot \overrightarrow{\Omega_{2/0}} + \overrightarrow{GA} \wedge m_2 \overrightarrow{V_{G\in 2/0}}$$

$$\boxed{\overrightarrow{\sigma_G(2/0)} = \begin{pmatrix} [(A_2 - m_2 \ell^2)\sin\beta - (F_2 + m_2 d\ell)\cos\beta]\dot\alpha \\ [(B_2 - m_2 d^2)\cos\beta - (F_2 + m_2 d\ell)\sin\beta]\dot\alpha \\ [C_2 - m_2(d^2 + \ell^2)]\dot\beta \end{pmatrix}_{R_2}}$$

2ᵉ méthode

Sachant que $\overrightarrow{AG} = \begin{pmatrix} -d \\ \ell \\ 0 \end{pmatrix}_{R_2}$ le théorème de Huygens donne :

$$\begin{pmatrix} A_2 & -F_2 & 0 \\ -F_2 & B_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{R_2} = \overline{\overline{J_G(2)}} + m_2 \begin{pmatrix} \ell^2 & \ell d & 0 \\ \ell d & d^2 & 0 \\ 0 & 0 & \ell^2 + d^2 \end{pmatrix}_{R_2}$$

Donc :

$$\overline{\overline{J_a(2)}} = \begin{pmatrix} A_2 - m_2 \ell^2 & -(F_2 + m_2 d\ell) & 0 \\ -(F_2 + m_2 d\ell) & B_2 - m_2 d^2 & 0 \\ 0 & 0 & C_2 - m_2(d^2+\ell^2) \end{pmatrix}_{R_2}$$

$$\overrightarrow{\sigma_a(2/0)} = \overline{\overline{J_a(2)}} \cdot \overrightarrow{\Omega_{2/0}} = \begin{pmatrix} A_2 - m_2 \ell^2 & -(F_2 + m_2 d\ell) & 0 \\ -(F_2 + m_2 d\ell) & B_2 - m_2 d^2 & 0 \\ 0 & 0 & C_2 - m_2(d^2+\ell^2) \end{pmatrix} \cdot \begin{pmatrix} \dot\alpha \sin\beta \\ \dot\alpha \cos\beta \\ \dot\beta \end{pmatrix}$$

$$\boxed{\overrightarrow{\sigma_a(2/0)} = \begin{pmatrix} \left[(A_2 - m_2 \ell^2)\sin\beta - (F_2 + m_2 d\ell)\cos\beta\right]\dot\alpha \\ \left[(B_2 - m_2 d^2)\cos\beta - (F_2 + m_2 d\ell)\sin\beta\right]\dot\alpha \\ \left[C_2 - m_2(d^2+\ell^2)\right]\dot\beta \end{pmatrix}_{R_2}}$$

4) $\vec{OA} + \vec{AC} + \vec{CO} = \vec{0} \iff -r\vec{X_1} + x_c\vec{X_2} + y_c\vec{Y_2} - x\vec{X_1} - y\vec{Y_1} = \vec{0}$

$\iff -r\vec{X_1} + x_c\cos\beta\,\vec{X_1} + x_c\sin\beta\,\vec{Y_1} - y_c\sin\beta\,\vec{X_1} + y_c\cos\beta\,\vec{Y_1} - x\vec{X_1} - y\vec{Y_1} = \vec{0}$

soit en projection sur $\vec{Y_1}$

$$x_c\sin\beta + y_c\cos\beta - y = 0$$

Donc

et :

$$\boxed{\begin{aligned} y &= x_c\sin\beta + y_c\cos\beta \\ \dot y &= (x_c\cos\beta - y_c\sin\beta)\,\dot\beta \end{aligned}}$$

$$\boxed{\overrightarrow{V_{G_3\in 3/0}} = \left(\frac{d\,\overrightarrow{OG_3}}{dt}\right)_{R_0} = \left(\frac{d\,(y+a_3)\vec{Y_0}}{dt}\right)_{R_0} = \dot\beta(x_c\cos\beta - y_c\sin\beta)\,\vec{Y_1}}$$

5) Calcul de l'energie cinetique de S/R_0

$$E_c(1/0) = \frac{1}{2}\,\overrightarrow{\Omega_{1/0}} \cdot \overline{\overline{J_O(1)}} \cdot \overrightarrow{\Omega_{1/0}} = \begin{pmatrix} 0 \\ \dot\alpha \\ 0 \end{pmatrix} \cdot \begin{pmatrix} A_1 & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & C_1 \end{pmatrix} \begin{pmatrix} 0 \\ \dot\alpha \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \dot\alpha B_1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \dot\alpha \\ 0 \end{pmatrix} = B_1\dot\alpha^2$$

$$E_c(3/0) = \frac{1}{2} m_3 \overrightarrow{V_{G_3\in 3/0}}^2 + \frac{1}{2}\overrightarrow{\Omega_{3/0}}\,\overline{\overline{J_{G_3}(3)}}\,\overrightarrow{\Omega_{3/0}} \quad \text{avec } \overrightarrow{\Omega_{3/0}} = \overrightarrow{\Omega_{1/0}} = \dot\alpha\,\vec{Y_1}$$

$$E_c(3/0) = \frac{1}{2}\left[m_3(x_c\cos\beta - y_c\sin\beta)^2\dot\beta^2 + B_3\dot\alpha^2\right]$$

$$E_c(2/0) = \{\mathcal{C}(2/0)\} \otimes \{\mathcal{V}(2/0)\} = {}_G\left\{\begin{matrix} m_2 \overrightarrow{V_{G\in 2/0}} \\ \overrightarrow{\sigma_G(2/0)}\end{matrix}\right\} \otimes {}_G\left\{\begin{matrix}\overrightarrow{\Omega_{2/0}} \\ \overrightarrow{V_{G\in 2/0}}\end{matrix}\right\}$$

$$E_c(2/0) = \frac{1}{2} m_2 \overrightarrow{V_{G\in 2/0}}^2 + \frac{1}{2} \overrightarrow{\Omega(2/0)} \cdot \overrightarrow{\sigma_G(2/0)}$$

$$= \frac{1}{2} m_2\left[(\ell^2+d^2)\dot\beta^2 + (r+d\cos\beta+\ell\sin\beta)\dot\alpha^2\right]$$

$$+\frac{1}{2}\begin{pmatrix}\dot\alpha\sin\beta\\ \dot\alpha\cos\beta\\ \dot\beta\end{pmatrix}\cdot\begin{pmatrix}\left[(A_2-m_2\ell^2)\sin\beta-(F_2+m_2 d\ell)\cos\beta\right]\dot\alpha \\ \left[(B_2-m_2 d^2)\cos\beta-(F_2+m_2 d\ell)\sin\beta\right]\dot\alpha \\ \left[C_2-m_2(d^2+\ell^2)\right]\dot\beta\end{pmatrix}$$

$$E_c(2/0) = \frac{1}{2}\left[(A_2-m_2\ell^2)\sin^2\beta+(B_2-m_2 d^2)\cos^2\beta-2(F_2+m_2 d\ell)\cos\beta\sin\beta + m_2(r+d\cos\beta+\ell\sin\beta)^2\right]\dot\alpha^2+\frac{1}{2} C_2\dot\beta^2$$

$$E_c(S/0) = E_c(1/0) + 2E_c(2/0) + E_c(3/0)$$

$$\boxed{E_c(S/0) = \frac{1}{2}\left[B_1+B_3+m_3(x_c\cos\beta-y_c\sin\beta)^2+2(A_2-m_2\ell^2)\sin^2\beta+2(B_2-m_2\ell^2)\cos^2\beta - 4(F_2+m_2 d\ell)\cos\beta\sin\beta\right]\dot\alpha^2+\frac{1}{2}\left[2C_2+m_3(x_c\cos\beta-y_c\sin\beta)^2\right]\dot\beta^2}$$