Sommes, produits et trigonométrie

1 Calculer $\dfrac{\binom{10}{7}}{\binom{10}{4}} = \dfrac{4}{7}$

2 Pour $n \in \mathbb{N}^*$, calculer $\displaystyle\prod_{1\leqslant i,\, j\leqslant n} ij = (n!)^{2n}$

3 $\displaystyle\sum_{1\leqslant i\leqslant j\leqslant n} ij = \frac{n(n+1)(n+2)(3n+1)}{24}$

4 Développer $(-x+2y)^5 = -x^5 + 10x^4y - 40x^3y^2 + 80x^2y^3 - 80xy^4 + 32y^5$

5 Soit $n \in \mathbb{N}$, compléter $(a+b)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} a^k\, b^{n-k}$.

6 a Résoudre sur $[0\,;2\pi]$, $\sin\left(2x - \frac{\pi}{4}\right) = \cos(x)$.

$$\begin{aligned}
\sin\left(2x-\frac{\pi}{4}\right) = \cos(x) &\iff \sin\left(2x-\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{2}-x\right)\\
&\iff 2x - \frac{\pi}{4} \equiv \frac{\pi}{2} - x\ [2\pi] \quad \text{ou} \quad 2x - \frac{\pi}{4} \equiv \pi - \frac{\pi}{2} + x\ [2\pi]\\
&\iff x \equiv \frac{\pi}{4}\left[\frac{2\pi}{3}\right] \quad \text{ou} \quad x \equiv \frac{3\pi}{4}\ [2\pi]
\end{aligned}$$

$$\mathscr{S} = \left\{\frac{\pi}{4} + \frac{2k\pi}{3},\ k\in\mathbb{Z}\right\} \cup \left\{\frac{3\pi}{4} + 2k\pi,\ k\in\mathbb{Z}\right\}.$$

Commentaires : *Attention à bien diviser le modulo 2π aussi !*

$$\mathscr{S}_{|[0;2\pi]} = \left\{\frac{\pi}{4};\ \frac{3\pi}{4};\ \frac{11\pi}{12};\ \frac{19\pi}{12}\right\}.$$

7 Simplifier $\cos(2x)\cos(x) + \sin(2x)\sin(x) = \cos(x)$.

8 En remarquant que $\frac{1}{12} = \frac{1}{3} - \frac{1}{4}$, donner la valeur exacte de $\cos\left(\frac{\pi}{12}\right)$.

$$\begin{aligned}
\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}\\
&= \frac{\sqrt{2}+\sqrt{6}}{4}.
\end{aligned}$$

9 Donner la valeur exacte de $\cos\left(\frac{\pi}{8}\right)$.

$$\begin{aligned}
\cos\left(\frac{\pi}{4}\right) &= \cos\left(2 \times \frac{\pi}{8}\right) = 2\cos^2\left(\frac{\pi}{8}\right) - 1 \\
\cos\left(\frac{\pi}{8}\right) &= +\sqrt{\frac{1+\cos\left(\frac{\pi}{4}\right)}{2}} \qquad \textit{car } \frac{\pi}{8} \in \left[-\frac{\pi}{2}\,;-\frac{\pi}{2}\right] \\
&= \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2+\sqrt{2}}}{2}.
\end{aligned}$$

10 En déduire la valeur exacte de $\cos\left(\frac{3\pi}{8}\right)$.

$$\cos\left(\frac{3\pi}{8}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{8}\right) = \sin\left(\frac{\pi}{8}\right) = \sqrt{1-\cos^2\left(\frac{\pi}{8}\right)} = \frac{\sqrt{2-\sqrt{2}}}{2}, \quad \textit{car } \frac{\pi}{8} \in [0\,;\pi]\,.$$

Sommes, produits et trigonométrie

1 Calculer $\dfrac{\binom{10}{3}}{\binom{12}{3}} = \dfrac{6}{11}$

2 Pour $n \in \mathbb{N}^*$, calculer $\displaystyle\prod_{1 \leqslant i,\, j \leqslant n} i^j = (n!)^{\frac{n(n+1)}{2}}$

3 $\displaystyle\sum_{1 \leqslant i \leqslant j \leqslant n} \frac{i}{j+1} = \frac{n(n+1)}{4}$

4 Développer $\left(\dfrac{1}{3} + y\right)^4 = \dfrac{1}{81} + \dfrac{4}{27}y + \dfrac{2}{3}y^2 + \dfrac{4}{3}y^3 + y^4$

5 Soit $n \in \mathbb{N}^*$, compléter $a^n - b^n = (a-b)\displaystyle\sum_{k=0}^{n-1} a^k b^{n-1-k}$.

6 Résoudre sur $[0\,;2\pi]$, $\cos\left(2x + \dfrac{\pi}{3}\right) = \sin(x)$.

$$\begin{aligned}
\cos\left(2x + \frac{\pi}{3}\right) = \sin(x) &\iff \cos\left(2x + \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{2} - x\right) \\
&\iff 2x + \frac{\pi}{3} \equiv \frac{\pi}{2} - x \ [2\pi] \quad \text{ou} \quad 2x + \frac{\pi}{3} \equiv -\frac{\pi}{2} + x \ [2\pi] \\
&\iff x \equiv \frac{\pi}{18} \left[\frac{2\pi}{3}\right] \quad \text{ou} \quad x \equiv -\frac{5\pi}{6} \ [2\pi]
\end{aligned}$$

$$\mathscr{S} = \left\{\frac{\pi}{18} + \frac{2k\pi}{3},\, k \in \mathbb{Z}\right\} \cup \left\{-\frac{5\pi}{6} + 2k\pi,\, k \in \mathbb{Z}\right\}.$$

Commentaires : *Attention à bien diviser le modulo 2π aussi !*

$$\mathscr{S}_{|[0;2\pi]} = \left\{\frac{\pi}{18}; \frac{13\pi}{18}; \frac{25\pi}{18}; \frac{7\pi}{6}\right\}.$$

7 Simplifier $\sin(2x)\cos(x) - \cos(2x)\sin(x) = \sin(x)$.

8 En remarquant que $\dfrac{1}{12} = \dfrac{1}{3} - \dfrac{1}{4}$, donner la valeur exacte de $\sin\left(\dfrac{\pi}{12}\right)$.

$$\begin{aligned}
\sin\left(\frac{\pi}{12}\right) &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\
&= \frac{\sqrt{6} - \sqrt{2}}{4}.
\end{aligned}$$

9 Donner la valeur exacte de $\sin\left(\dfrac{\pi}{8}\right)$.

$$\begin{aligned}
\cos\left(\frac{\pi}{4}\right) &= \cos\left(2\times\frac{\pi}{8}\right) = 1 - 2\sin^2\left(\frac{\pi}{8}\right) \\
\sin\left(\frac{\pi}{8}\right) &= +\sqrt{\frac{1-\cos\left(\frac{\pi}{4}\right)}{2}} \qquad \textit{car } \frac{\pi}{8}\in[0\,;\pi] \\
&= \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2-\sqrt{2}}}{2}.
\end{aligned}$$

10 En déduire la valeur exacte de $\sin\left(\dfrac{5\pi}{8}\right)$.

$$\sin\left(\frac{5\pi}{8}\right) = \sin\left(\frac{\pi}{2}+\frac{\pi}{8}\right) = \cos\left(\frac{\pi}{8}\right) = \sqrt{1-\sin^2\left(\frac{\pi}{8}\right)} = \frac{\sqrt{2+\sqrt{2}}}{2}, \quad \textit{car } \frac{\pi}{8}\in\left[-\frac{\pi}{2}\,;\frac{\pi}{2}\right].$$