Développements usuels en 0

$\dfrac{1}{1-x} \underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} x^k + \mathrm{o}(x^n) \underset{x\to 0}{=} 1 + x + x^2 + \dots x^n + \mathrm{o}(x^n).$
$\ln(1-x) \underset{x\to 0}{=}$	$\displaystyle -\sum_{k=1}^{n} \frac{x^k}{k} + \mathrm{o}(x^n) \underset{x\to 0}{=} -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots - \frac{x^n}{n} + \mathrm{o}(x^n).$
$\dfrac{1}{1+x} \underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} (-1)^k x^k + \mathrm{o}(x^n) \underset{x\to 0}{=} 1 - x + x^2 + \dots (-1)^n x^n + \mathrm{o}(x^n).$
$\ln(1+x) \underset{x\to 0}{=}$	$\displaystyle\sum_{k=1}^{n} (-1)^{k+1} \frac{x^k}{k} + \mathrm{o}(x^n) \underset{x\to 0}{=} x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + (-1)^{n+1}\frac{x^n}{n} + \mathrm{o}(x^n).$
$\arctan(x) \underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1} + \mathrm{o}(x^{2n+2}) \underset{x\to 0}{=} x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \mathrm{o}(x^{2n+2}).$
$(1+x)^\alpha \underset{x\to 0}{=}$	$\displaystyle 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \dots + \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}{n!}x^n + \mathrm{o}(x^n) \quad .$
$\underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} \binom{\alpha}{k}_{\mathbb{R}} x^k + \mathrm{o}(x^n)$ où $\binom{\alpha}{k}_{\mathbb{R}}$ est le coefficient du binôme généralisé à $\alpha \in \mathbb{R}$
$\sqrt{1+x} \underset{x\to 0}{=}$	$\displaystyle 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots + (-1)^{n-1}\frac{1\times 3\times \dots \times (2n-3)}{2\times 4\times \dots \times 2n} x^n + \mathrm{o}(x^n).$
$\mathrm{e}^x \underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} \frac{x^k}{k!} + \mathrm{o}(x^n) \underset{x\to 0}{=} 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots + \frac{x^n}{n!} + \mathrm{o}(x^n).$
$\cos(x) \underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} + \mathrm{o}(x^{2n+1}) \underset{x\to 0}{=} 1 - \frac{x^2}{2} + \frac{x^4}{24} + \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \mathrm{o}(x^{2n+1}).$
$\sin(x) \underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \mathrm{o}(x^{2n+2}) \underset{x\to 0}{=} x - \frac{x^3}{6} + \frac{x^5}{120} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \mathrm{o}(x^{2n+2}).$
$\mathrm{ch}(x) \underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} + \mathrm{o}(x^{2n+1}) \underset{x\to 0}{=} 1 + \frac{x^2}{2} + \frac{x^4}{24} + \dots + \frac{x^{2n}}{(2n)!} + \mathrm{o}(x^{2n+1}).$
$\mathrm{sh}(x) \underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!} + \mathrm{o}(x^{2n+2}) \underset{x\to 0}{=} x + \frac{x^3}{6} + \frac{x^5}{120} + \dots + \frac{x^{2n+1}}{(2n+1)!} + \mathrm{o}(x^{2n+2}).$
$\tan(x) \underset{x\to 0}{=}$	$\displaystyle x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \mathrm{o}(x^8).$

Exercice 1 : Effectuer le DL au voisinage de 0 à l'ordre indiqué de la fonction suivante :

1. $\tan^2(x)$ — $n = 7$
2. $\dfrac{\ln(1+x)}{1+x}$ — $n = 4$
3. $e^{x\sin(x)}$ — $n = 6$
4. $\sqrt{1+\sin(x)}$ — $n = 3$
5. $\sqrt{\cos(x)}$ — $n = 7$
6. $\dfrac{1}{\cos(x)}$ — $n = 5$
7. $\dfrac{x}{\sin(x)}$ — $n = 5$
8. $\ln\left(\dfrac{\sin(x)}{x}\right)$ — $n = 5$
9. $\ln\left(\dfrac{\arctan(x)}{x}\right)$ — $n = 7$
10. $(1+x)^x$ — $n = 5$
11. $(1+x)^{\frac{1}{x}}$ — $n = 3$
12. $\ln\left(\ln\left((1+x)^{\frac{1}{x}}\right)\right)$ — $n = 4$
13. $\left(\dfrac{\sin(x)}{x}\right)^{\frac{3}{x^2}}$ — $n = 3$
14. $(\cos(x))^p$ — $n = 6$
15. $\left(\dfrac{1+e^x}{2}\right)^p$ — $n = 2$
16. $\left(\dfrac{\sin(x)}{x}\right)^p$ — $n = 5$

Correction :

1. $\tan^2(x) = x^2 + \frac{2}{3}x^4 + \frac{17}{45}x^6 + \underset{x\to 0}{o}(x^7)$

2. $\frac{\ln(1+x)}{1+x} = x - \frac{3}{2}x^2 + \frac{11}{6}x^3 - \frac{25}{12}x^4 + \underset{x\to 0}{o}(x^4)$

3. $e^{x\sin(x)} = 1 + x^2 + \frac{1}{3}x^4 + \frac{1}{120}x^6 + \underset{x\to 0}{o}(x^6)$

4. $\sqrt{1+\sin(x)} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{48}x^3 + \underset{x\to 0}{o}(x^3)$

5. $\sqrt{\cos(x)} = 1 - \frac{1}{4}x^2 - \frac{1}{96}x^4 - \frac{19}{5760}x^6 + \underset{x\to 0}{o}(x^7)$

6. $\frac{1}{\cos(x)} = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \underset{x\to 0}{o}(x^5)$

7. $\frac{x}{\sin(x)} = 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \underset{x\to 0}{o}(x^5)$

8. $\ln\left(\frac{\sin(x)}{x}\right) = -\frac{1}{6}x^2 - \frac{1}{180}x^4 + \underset{x\to 0}{o}(x^5)$

9. $\ln\left(\frac{\arctan(x)}{x}\right) = -\frac{1}{3}x^2 + \frac{13}{90}x^4 - \frac{251}{2835}x^6 + \underset{x\to 0}{o}(x^7)$

10. $(1+x)^x = 1 + x^2 - \frac{1}{2}x^3 + \frac{5}{6}x^4 - \frac{3}{4}x^5 + \underset{x\to 0}{o}(x^5)$

11. $(1+x)^{\frac{1}{x}} = e \times \left[1 - \frac{1}{2}x + \frac{11}{24}x^2 - \frac{7}{16}x^3\right] + \underset{x\to 0}{o}(x^3)$

12. $\ln\left(\ln\left((1+x)^{\frac{1}{x}}\right)\right) = -\frac{1}{2}x + \frac{5}{24}x^2 - \frac{1}{8}x^3 + \frac{251}{2880}x^4 + \underset{x\to 0}{o}(x^4)$

13. $\left(\frac{\sin(x)}{x}\right)^{\frac{3}{x^2}} = \frac{1}{\sqrt{e}} \times \left[1 - \frac{1}{60}x^2\right] + \underset{x\to 0}{o}(x^3)$

14. $(\cos(x))^p = 1 - \frac{p}{2}x^2 + \frac{p(3p-2)}{24}x^4 - \frac{p[15(p-1)^2+1]}{720}x^6 + \underset{x\to 0}{o}(x^6)$

15. $\left(\frac{1+e^x}{2}\right)^p = 1 + \frac{p}{2}x + \frac{p\,(p+1)}{8}x^2 + \underset{x\to 0}{o}(x^2)$

16. $\left(\frac{\sin(x)}{x}\right)^p = 1 - \frac{p}{6}x^2 + \frac{p\,(5p-2)}{360}x^4 + \underset{x\to 0}{o}(x^5)$

Exercice 2 (Applications) : Calculer les limites suivantes :

1. $\displaystyle\lim_{x\to0}\frac{\sin(\sin(x)-x)}{\sqrt{1+x^3}-1}$

2. $\displaystyle\lim_{x\to0}\frac{3\tan 4x-4\tan 3x}{3\sin 4x-4\sin 3x}$

3. $\displaystyle\lim_{x\to0}\frac{x(e^x+1)-2(e^x-1)}{x^3}$

4. $\displaystyle\lim_{x\to0}\frac{\ln(1+x)-\ln(1-x)}{\arctan(1+x)-\arctan(1-x)}$

5. $\displaystyle\lim_{x\to0}\left(\frac{1}{x}-\frac{1}{\ln(1+x)}\right)$

6. $\displaystyle\lim_{x\to0}\frac{(1+x)^{\frac{1}{x}}-e}{x}$

7. $\displaystyle\lim_{x\to1}\frac{1-x+\ln(x)}{1-\sqrt{2x-x^2}}$

8. $\displaystyle\lim_{x\to1}\left(\frac{1}{2(1-\sqrt{x})}-\frac{1}{3(1-\sqrt[3]{x})}\right)$

Correction :

1

◇ $\sin(x)-x\underset{x\to0}{\longrightarrow}0$ donc $\sin(\sin(x)-x)\underset{x\to0}{\sim}\sin(x)-x$.

$\sin(x)-x=\left[x-\dfrac{x^3}{3!}+\underset{x\to0}{o}(x^3)\right]-x=-\dfrac{x^3}{3!}+\underset{x\to0}{o}(x^3)\underset{x\to0}{\sim}-\dfrac{x^3}{3!}$, d'où

$$\sin(\sin(x)-x)\underset{x\to0}{\sim}-\frac{x^3}{3!}$$

◇ Comme $x^3\underset{x\to0}{\longrightarrow}0$, $\sqrt{1+x^3}-1\underset{x\to0}{\sim}\dfrac{1}{2}x^3$.

Donc $\dfrac{\sin(\sin(x)-x)}{\sqrt{1+x^3}-1}\underset{x\to0}{\sim}\dfrac{-\frac{x^3}{3!}}{\frac{1}{2}x^3}\underset{x\to0}{\sim}-\dfrac{1}{3}$ et $\displaystyle\lim_{x\to0}\frac{\sin(\sin(x)-x)}{\sqrt{1+x^3}-1}=-\frac{1}{3}$.

2

◇ $3\tan 4x-4\tan 3x=3\left[4x+\dfrac{(4x)^3}{3}+\underset{x\to0}{o}(x^3)\right]-4\left[3x+\dfrac{(3x)^3}{3}+\underset{x\to0}{o}(x^3)\right]$
$=64x^3-36x^3+\underset{x\to0}{o}(x^3)\underset{x\to0}{\sim}28x^3$

◇ $3\sin 4x-4\sin 3x=3\left[4x-\dfrac{(4x)^3}{3!}+\underset{x\to0}{o}(x^3)\right]-4\left[3x-\dfrac{(3x)^3}{3!}+\underset{x\to0}{o}(x^3)\right]$
$=-32x^3+18x^3+\underset{x\to0}{o}(x^3)\underset{x\to0}{\sim}-14x^3$

Donc $\dfrac{3\tan 4x-4\tan 3x}{3\sin 4x-4\sin 3x}\underset{x\to0}{\sim}\dfrac{28x^3}{-14x^3}\underset{x\to0}{\sim}-2$ et $\displaystyle\lim_{x\to0}\frac{3\tan 4x-4\tan 3x}{3\sin 4x-4\sin 3x}=-2$.

3

$$\frac{x(e^x+1)-2(e^x-1)}{x^3}=\frac{x\left[1+x+\frac{x^2}{2}+\underset{x\to0}{o}(x^2)+1\right]-2\left[1+x+\frac{x^2}{2}+\frac{x^3}{6}+\underset{x\to0}{o}(x^3)-1\right]}{x^3}$$
$$=\frac{\left[2x+x^2+\frac{x^3}{2}+\underset{x\to0}{o}(x^3)\right]-\left[2x+x^2+\frac{x^3}{3}+\underset{x\to0}{o}(x^3)\right]}{x^3}$$
$$=\frac{\frac{1}{6}x^3+\underset{x\to0}{o}(x^3)}{x^3}=\frac{1}{6}+\underset{x\to0}{o}(1)$$
et $\displaystyle\lim_{x\to0}\frac{x(e^x+1)-2(e^x-1)}{x^3}=\frac{1}{6}$.

4

◇ $\ln(1+x)-\ln(1-x)=\left[x+\underset{x\to0}{o}(x)\right]-\left[-x+\underset{x\to0}{o}(x)\right]=2x+\underset{x\to0}{o}(x)\underset{x\to0}{\sim}2x$.

◇ $f(x)=\arctan(1+x)$. On a $f'(x)=\dfrac{1}{1+(1+x)^2}$ donc $f'(0)=\dfrac{1}{2}$.

Le $DL_1(0)$ de f est donné par $f(x)=f(0)+f'(0)x+\underset{x\to0}{o}(x)$

$$\begin{cases}\arctan(1+x)=\dfrac{\pi}{4}+\dfrac{1}{2}x+\underset{x\to0}{o}(x)\\[2mm]\arctan(1-x)=\dfrac{\pi}{4}-\dfrac{1}{2}x+\underset{x\to0}{o}(x)\end{cases}$$

Donc $\arctan(1+x)-\arctan(1-x)=x+\underset{x\to0}{o}(x)\underset{x\to0}{\sim}x$.

D'où,

$$\lim_{x\to 0} \frac{\ln(1+x)-\ln(1-x)}{\arctan(1+x)-\arctan(1-x)} = 2.$$

5 $\dfrac{1}{x} - \dfrac{1}{\ln(1+x)} = \dfrac{\ln(1+x)-x}{x\ln(1+x)}$

$$= \frac{\left[x - \frac{x^2}{2} + \underset{x\to 0}{o}(x^2)\right] - x}{x\ln(1+x)} = \frac{-\frac{x^2}{2} + \underset{x\to 0}{o}(x^2)}{x\ln(1+x)}$$

$$\underset{x\to 0}{\sim} \frac{-\frac{x^2}{2}}{x^2} \underset{x\to 0}{\sim} -\frac{1}{2} \quad \text{et} \quad \lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\ln(1+x)}\right) = -\frac{1}{2}.$$

6 $(1+x)^{\frac{1}{x}} = e^{\frac{1}{x}\ln(1+x)} = e^{\frac{1}{x}\left[x - \frac{x^2}{2} + \underset{x\to 0}{o}(x^2)\right]} = e^{1-\frac{x}{2}+\underset{x\to 0}{o}(x)} = e \times e^{-\frac{x}{2}+\underset{x\to 0}{o}(x)}$

$$= e\left[1 - \frac{x}{2} + \underset{x\to 0}{o}(x)\right] = e - \frac{e}{2}x + \underset{x\to 0}{o}(x).$$

Donc, $\dfrac{(1+x)^{\frac{1}{x}} - e}{x} = \dfrac{e - \frac{e}{2}x + \underset{x\to 0}{o}(x) - e}{x} = -\dfrac{e}{2} + \underset{x\to 0}{o}(1)$ et $\displaystyle\lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x} = -\frac{e}{2}.$

7 On pose $x = 1+h$.

$$\frac{1-x+\ln(x)}{1-\sqrt{2x-x^2}} = \frac{1-(1+h)+\ln(1+h)}{1-\sqrt{2(1+h)-(1+h)^2}} = \frac{-h + \left[h - \frac{h^2}{2} + \underset{h\to 0}{o}(h^2)\right]}{1-\sqrt{1-h^2}} \underset{h\to 0}{\sim} \frac{-\frac{h^2}{2}}{\frac{1}{2}h^2} \underset{h\to 0}{\sim} -1.$$

Donc,

$$\lim_{x\to 1} \frac{1-x+\ln(x)}{1-\sqrt{2x-x^2}} = -1.$$

8

$$\frac{1}{2(1-\sqrt{x})} - \frac{1}{3(1-\sqrt[3]{x})} = \frac{3(1-\sqrt[3]{x}) - 2(1-\sqrt{x})}{6(1-\sqrt{x})(1-\sqrt[3]{x})}$$

On pose $x = 1+h$.

- $3(1-\sqrt[3]{x}) - 2(1-\sqrt{x}) = 3\left[1 - \left(1 + \frac{1}{3}h - \frac{1}{9}h^2 + \underset{h\to 0}{o}(h^2)\right)\right] - 2\left[1 - \left(1 + \frac{1}{2}h - \frac{1}{8}h^2 + \underset{h\to 0}{o}(h^2)\right)\right]$

$$= \frac{1}{12}h^2 + \underset{h\to 0}{o}(h^2) \underset{h\to 0}{\sim} \frac{1}{12}h^2.$$

- et $6(1-\sqrt{x})(1-\sqrt[3]{x}) = 6\left(1-\sqrt{1+h}\right)\left(1-\sqrt[3]{1+h}\right) \underset{h\to 0}{\sim} 6\left(\frac{-1}{2}h\right)\left(\frac{-1}{3}h\right) \underset{h\to 0}{\sim} h^2.$

Donc,

$$\frac{1}{2(1-\sqrt{x})} - \frac{1}{3(1-\sqrt[3]{x})} \underset{h\to 0}{\sim} \frac{\frac{1}{12}h^2}{h^2} \underset{h\to 0}{\sim} \frac{1}{12} \quad \text{et} \quad \lim_{x\to 1}\left(\frac{1}{2(1-\sqrt{x})} - \frac{1}{3(1-\sqrt[3]{x})}\right) = \frac{1}{12}.$$