Développements usuels en 0

$\dfrac{1}{1-x}$	$\underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} x^k + o(x^n) \underset{x\to 0}{=} 1+x+x^2+\dots x^n + o(x^n)$.
$\ln(1-x)$	$\underset{x\to 0}{=}$	$\displaystyle -\sum_{k=1}^{n} \frac{x^k}{k} + o(x^n) \underset{x\to 0}{=} -x-\frac{x^2}{2}-\frac{x^3}{3}-\dots-\frac{x^n}{n}+o(x^n)$.
$\dfrac{1}{1+x}$	$\underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} (-1)^k x^k + o(x^n) \underset{x\to 0}{=} 1-x+x^2+\dots(-1)^n x^n + o(x^n)$.
$\ln(1+x)$	$\underset{x\to 0}{=}$	$\displaystyle\sum_{k=1}^{n} (-1)^{k+1}\frac{x^k}{k} + o(x^n) \underset{x\to 0}{=} x-\frac{x^2}{2}+\frac{x^3}{3}+\dots+(-1)^{n+1}\frac{x^n}{n}+o(x^n)$.
$\arctan(x)$	$\underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} (-1)^k\frac{x^{2k+1}}{2k+1} + o(x^{2n+2}) \underset{x\to 0}{=} x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots+(-1)^n\frac{x^{2n+1}}{2n+1}+o(x^{2n+2})$.
$(1+x)^\alpha$	$\underset{x\to 0}{=}$	$\displaystyle 1+\alpha x+\frac{\alpha(\alpha-1)}{2}x^2+\dots+\frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}{n!}x^n+o(x^n)$.
	$\underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} \binom{\alpha}{k}_{\mathbb{R}} x^k + o(x^n)$ où $\binom{\alpha}{k}_{\mathbb{R}}$ est le coefficient du binôme généralisé à $\alpha\in\mathbb{R}$
$\sqrt{1+x}$	$\underset{x\to 0}{=}$	$\displaystyle 1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\dots+(-1)^{n-1}\frac{1\times 3\times\dots\times(2n-3)}{2\times 4\times\dots\times 2n}x^n+o(x^n)$.
$\dfrac{1}{\sqrt{1+x}}$	$\underset{x\to 0}{=}$	$\displaystyle 1-\frac{1}{2}x+\frac{3}{8}x^2+\dots+(-1)^n\frac{1\times 3\times\dots\times(2n-1)}{2\times 4\times\dots\times 2n}x^n+o(x^n)$.
e^x	$\underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n) \underset{x\to 0}{=} 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\dots+\frac{x^n}{n!}+o(x^n)$.
$\cos(x)$	$\underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} + o(x^{2n+1}) \underset{x\to 0}{=} 1-\frac{x^2}{2}+\frac{x^4}{24}+\dots+\frac{(-1)^n x^{2n}}{(2n)!}+o(x^{2n+1})$.
$\sin(x)$	$\underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + o(x^{2n+2}) \underset{x\to 0}{=} x-\frac{x^3}{6}+\frac{x^5}{120}+\dots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+o(x^{2n+2})$.
$\mathrm{ch}(x)$	$\underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} + o(x^{2n+1}) \underset{x\to 0}{=} 1+\frac{x^2}{2}+\frac{x^4}{24}+\dots+\frac{x^{2n}}{(2n)!}+o(x^{2n+1})$.
$\mathrm{sh}(x)$	$\underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+2}) \underset{x\to 0}{=} x+\frac{x^3}{6}+\frac{x^5}{120}+\dots+\frac{x^{2n+1}}{(2n+1)!}+o(x^{2n+2})$.
$\tan(x)$	$\underset{x\to 0}{=}$	$\displaystyle x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+o(x^8)$.

Exercice 1 : Effectuer le DL au voisinage de 0 à l'ordre indiqué de la fonction suivante :

1. $\tan^2(x)$ $n=7$
2. $\dfrac{\ln(1+x)}{1+x}$ $n=4$
3. $e^{x\sin(x)}$ $n=6$
4. $\sqrt{1+\sin(x)}$ $n=3$
5. $\sqrt{\cos(x)}$ $n=7$
6. $\dfrac{1}{\cos(x)}$ $n=5$
7. $\dfrac{x}{\sin(x)}$ $n=5$
8. $\ln\left(\dfrac{\sin(x)}{x}\right)$ $n=5$
9. $\ln\left(\dfrac{\arctan(x)}{x}\right)$ $n=7$
10. $(1+x)^x$ $n=5$
11. $(1+x)^{\frac{1}{x}}$ $n=3$
12. $\ln\left(\ln\left((1+x)^{\frac{1}{x}}\right)\right)$ $n=4$
13. $\left(\dfrac{\sin(x)}{x}\right)^{\frac{3}{x^2}}$ $n=3$
14. $(\cos(x))^p$ $n=6$
15. $\left(\dfrac{1+e^x}{2}\right)^p$ $n=2$
16. $\left(\dfrac{\sin(x)}{x}\right)^p$ $n=5$

Développements usuels en 0

Correction :

1. $\tan^2(x) = x^2 + \frac{2}{3}x^4 + \frac{17}{45}x^6 + \underset{x\to 0}{o}(x^7)$

2. $\frac{\ln(1+x)}{1+x} = x - \frac{3}{2}x^2 + \frac{11}{6}x^3 - \frac{25}{12}x^4 + \underset{x\to 0}{o}(x^4)$

3. $e^{x\sin(x)} = 1 + x^2 + \frac{1}{3}x^4 + \frac{1}{120}x^6 + \underset{x\to 0}{o}(x^6)$

4. $\sqrt{1+\sin(x)} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{48}x^3 + \underset{x\to 0}{o}(x^3)$

5. $\sqrt{\cos(x)} = 1 - \frac{1}{4}x^2 - \frac{1}{96}x^4 - \frac{19}{5760}x^6 + \underset{x\to 0}{o}(x^7)$

6. $\frac{1}{\cos(x)} = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \underset{x\to 0}{o}(x^5)$

7. $\frac{x}{\sin(x)} = 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \underset{x\to 0}{o}(x^5)$

8. $\ln\left(\frac{\sin(x)}{x}\right) = -\frac{1}{6}x^2 - \frac{1}{180}x^4 + \underset{x\to 0}{o}(x^5)$

9. $\ln\left(\frac{\arctan(x)}{x}\right) = -\frac{1}{3}x^2 + \frac{13}{90}x^4 - \frac{251}{2835}x^6 + \underset{x\to 0}{o}(x^7)$

10. $(1+x)^x = 1 + x^2 - \frac{1}{2}x^3 + \frac{5}{6}x^4 - \frac{3}{4}x^5 + \underset{x\to 0}{o}(x^5)$

11. $(1+x)^{\frac{1}{x}} = e \times \left[1 - \frac{1}{2}x + \frac{11}{24}x^2 - \frac{7}{16}x^3\right] + \underset{x\to 0}{o}(x^3)$

12. $\ln\left(\ln\left((1+x)^{\frac{1}{x}}\right)\right) = -\frac{1}{2}x + \frac{5}{24}x^2 - \frac{1}{8}x^3 + \frac{251}{2880}x^4 + \underset{x\to 0}{o}(x^4)$

13. $\left(\frac{\sin(x)}{x}\right)^{\frac{3}{x^2}} = \frac{1}{\sqrt{e}} \times \left[1 - \frac{1}{60}x^2\right] + \underset{x\to 0}{o}(x^3)$

14. $(\cos(x))^p = 1 - \frac{p}{2}x^2 + \frac{p(3p-2)}{24}x^4 - \frac{p[15(p-1)^2+1]}{720}x^6 + \underset{x\to 0}{o}(x^6)$

15. $\left(\frac{1+e^x}{2}\right)^p = 1 + \frac{p}{2}x + \frac{p\,(p+1)}{8}x^2 + \underset{x\to 0}{o}(x^2)$

16. $\left(\frac{\sin(x)}{x}\right)^p = 1 - \frac{p}{6}x^2 + \frac{p\,(5p-2)}{360}x^4 + \underset{x\to 0}{o}(x^5)$

Exercice 2 (Applications) : Calculer les limites suivantes :

1. $\displaystyle\lim_{x\to 0} \frac{\sin(\sin(x)-x)}{\sqrt{1+x^3}-1}$
2. $\displaystyle\lim_{x\to 0} \frac{3\tan 4x - 4\tan 3x}{3\sin 4x - 4\sin 3x}$
3. $\displaystyle\lim_{x\to 0} \frac{x(e^x+1)-2(e^x-1)}{x^3}$
4. $\displaystyle\lim_{x\to 0} \frac{\ln(1+x)-\ln(1-x)}{\arctan(1+x)-\arctan(1-x)}$
5. $\displaystyle\lim_{x\to 0} \left(\frac{1}{x}-\frac{1}{\ln(1+x)}\right)$
6. $\displaystyle\lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}}-e}{x}$
7. $\displaystyle\lim_{x\to 1} \frac{1-x+\ln(x)}{1-\sqrt{2x-x^2}}$
8. $\displaystyle\lim_{x\to 1} \left(\frac{1}{2(1-\sqrt{x})}-\frac{1}{3(1-\sqrt[3]{x})}\right)$

Correction :

1. ◇ $\sin(x)-x \underset{x\to 0}{\longrightarrow} 0$ donc $\sin(\sin(x)-x) \underset{x\to 0}{\sim} \sin(x)-x$.

$$\sin(x)-x = \left[x-\frac{x^3}{3!}+\underset{x\to 0}{o}(x^3)\right]-x = -\frac{x^3}{3!}+\underset{x\to 0}{o}(x^3) \underset{x\to 0}{\sim} -\frac{x^3}{3!}, \text{ d'où}$$

$$\sin(\sin(x)-x) \underset{x\to 0}{\sim} -\frac{x^3}{3!}$$

◇ Comme $x^3 \underset{x\to 0}{\longrightarrow} 0$, $\sqrt{1+x^3}-1 \underset{x\to 0}{\sim} \frac{1}{2}x^3$.

Donc $\displaystyle\frac{\sin(\sin(x)-x)}{\sqrt{1+x^3}-1} \underset{x\to 0}{\sim} \frac{-\frac{x^3}{3!}}{\frac{1}{2}x^3} \underset{x\to 0}{\sim} -\frac{1}{3}$ et $\displaystyle\lim_{x\to 0}\frac{\sin(\sin(x)-x)}{\sqrt{1+x^3}-1} = -\frac{1}{3}$.

2. ◇ $$3\tan 4x - 4\tan 3x = 3\left[4x+\frac{(4x)^3}{3}+\underset{x\to 0}{o}(x^3)\right]-4\left[3x+\frac{(3x)^3}{3}+\underset{x\to 0}{o}(x^3)\right]$$
$$= 64x^3-36x^3+\underset{x\to 0}{o}(x^3) \underset{x\to 0}{\sim} 28x^3$$

◇ $$3\sin 4x - 4\sin 3x = 3\left[4x-\frac{(4x)^3}{3!}+\underset{x\to 0}{o}(x^3)\right]-4\left[3x-\frac{(3x)^3}{3!}+\underset{x\to 0}{o}(x^3)\right]$$
$$= -32x^3+18x^3+\underset{x\to 0}{o}(x^3) \underset{x\to 0}{\sim} -14x^3$$

Donc $\displaystyle\frac{3\tan 4x-4\tan 3x}{3\sin 4x-4\sin 3x} \underset{x\to 0}{\sim} \frac{28x^3}{-14x^3} \underset{x\to 0}{\sim} -2$ et $\displaystyle\lim_{x\to 0}\frac{3\tan 4x-4\tan 3x}{3\sin 4x-4\sin 3x} = -2$.

3. $$\frac{x(e^x+1)-2(e^x-1)}{x^3} = \frac{x\left[1+x+\frac{x^2}{2}+\underset{x\to 0}{o}(x^2)+1\right]-2\left[1+x+\frac{x^2}{2}+\frac{x^3}{6}+\underset{x\to 0}{o}(x^3)-1\right]}{x^3}$$
$$= \frac{\left[2x+x^2+\frac{x^3}{2}+\underset{x\to 0}{o}(x^3)\right]-\left[2x+x^2+\frac{x^3}{3}+\underset{x\to 0}{o}(x^3)\right]}{x^3}$$
$$= \frac{\frac{1}{6}x^3+\underset{x\to 0}{o}(x^3)}{x^3} = \frac{1}{6}+\underset{x\to 0}{o}(1) \quad \text{et} \quad \lim_{x\to 0}\frac{x(e^x+1)-2(e^x-1)}{x^3} = \frac{1}{6}.$$

4. ◇ $\ln(1+x)-\ln(1-x) = \left[x+\underset{x\to 0}{o}(x)\right]-\left[-x+\underset{x\to 0}{o}(x)\right] = 2x+\underset{x\to 0}{o}(x) \underset{x\to 0}{\sim} 2x$.

◇ $f(x)=\arctan(1+x)$. On a $f'(x)=\dfrac{1}{1+(1+x)^2}$ donc $f'(0)=\dfrac{1}{2}$.

Le $\mathrm{DL}_1(0)$ de f est donné par $f(x)=f(0)+f'(0)x+\underset{x\to 0}{\mathrm{o}}(x)$

$$\begin{cases}\arctan(1+x)=\dfrac{\pi}{4}+\dfrac{1}{2}x+\underset{x\to 0}{\mathrm{o}}(x)\\ \arctan(1-x)=\dfrac{\pi}{4}-\dfrac{1}{2}x+\underset{x\to 0}{\mathrm{o}}(x)\end{cases}$$

Donc $\arctan(1+x)-\arctan(1-x)=x+\underset{x\to 0}{\mathrm{o}}(x)\underset{x\to 0}{\sim}x$.

D'où,

$$\lim_{x\to 0}\frac{\ln(1+x)-\ln(1-x)}{\arctan(1+x)-\arctan(1-x)}=2.$$

5. $$\frac{1}{x}-\frac{1}{\ln(1+x)}=\frac{\ln(1+x)-x}{x\ln(1+x)}=\frac{\left[x-\frac{x^2}{2}+\underset{x\to 0}{\mathrm{o}}(x^2)\right]-x}{x\ln(1+x)}=\frac{-\frac{x^2}{2}+\underset{x\to 0}{\mathrm{o}}(x^2)}{x\ln(1+x)}\underset{x\to 0}{\sim}\frac{-\frac{x^2}{2}}{x^2}\underset{x\to 0}{\sim}-\frac{1}{2}$$ et $$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\ln(1+x)}\right)=-\frac{1}{2}.$$

6. $$(1+x)^{\frac{1}{x}}=e^{\frac{1}{x}\ln(1+x)}=e^{\frac{1}{x}\left[x-\frac{x^2}{2}+\underset{x\to 0}{\mathrm{o}}(x^2)\right]}=e^{1-\frac{x}{2}+\underset{x\to 0}{\mathrm{o}}(x)}=e\times e^{-\frac{x}{2}+\underset{x\to 0}{\mathrm{o}}(x)}=e\left[1-\frac{x}{2}+\underset{x\to 0}{\mathrm{o}}(x)\right]=e-\frac{e}{2}x+\underset{x\to 0}{\mathrm{o}}(x).$$

Donc, $\dfrac{(1+x)^{\frac{1}{x}}-e}{x}=\dfrac{e-\frac{e}{2}x+\underset{x\to 0}{\mathrm{o}}(x)-e}{x}=-\dfrac{e}{2}+\underset{x\to 0}{\mathrm{o}}(1)$ et $\displaystyle\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x}=-\frac{e}{2}$.

7. On pose $x=1+h$.

$$\frac{1-x+\ln(x)}{1-\sqrt{2x-x^2}}=\frac{1-(1+h)+\ln(1+h)}{1-\sqrt{2(1+h)-(1+h)^2}}=\frac{-h+\left[h-\frac{h^2}{2}+\underset{h\to 0}{\mathrm{o}}(h^2)\right]}{1-\sqrt{1-h^2}}\underset{h\to 0}{\sim}\frac{-\frac{h^2}{2}}{\frac{1}{2}h^2}\underset{h\to 0}{\sim}-1.$$

Donc,

$$\lim_{x\to 1}\frac{1-x+\ln(x)}{1-\sqrt{2x-x^2}}=-1.$$

8. $$\frac{1}{2(1-\sqrt{x})}-\frac{1}{3(1-\sqrt[3]{x})}=\frac{3(1-\sqrt[3]{x})-2(1-\sqrt{x})}{6(1-\sqrt{x})(1-\sqrt[3]{x})}$$

On pose $x=1+h$.

◇ $$3(1-\sqrt[3]{x})-2(1-\sqrt{x})=3\left[1-\left(1+\frac{1}{3}h-\frac{1}{9}h^2+\underset{h\to 0}{\mathrm{o}}(h^2)\right)\right]-2\left[1-\left(1+\frac{1}{2}h-\frac{1}{8}h^2+\underset{h\to 0}{\mathrm{o}}(h^2)\right)\right]=\frac{1}{12}h^2+\underset{h\to 0}{\mathrm{o}}(h^2)\underset{h\to 0}{\sim}\frac{1}{12}h^2.$$

◇ et $6(1-\sqrt{x})(1-\sqrt[3]{x})=6\left(1-\sqrt{1+h}\right)\left(1-\sqrt[3]{1+h}\right)\underset{h\to 0}{\sim}6\left(\dfrac{-1}{2}h\right)\left(\dfrac{-1}{3}h\right)\underset{h\to 0}{\sim}h^2$.

Donc,

$$\frac{1}{2(1-\sqrt{x})}-\frac{1}{3(1-\sqrt[3]{x})}\underset{h\to 0}{\sim}\frac{\frac{1}{12}h^2}{h^2}\underset{h\to 0}{\sim}\frac{1}{12}$$ et $$\lim_{x\to 1}\left(\frac{1}{2(1-\sqrt{x})}-\frac{1}{3(1-\sqrt[3]{x})}\right)=\frac{1}{12}.$$