Développements usuels en 0

$\dfrac{1}{1-x} \underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} x^k + \mathrm{o}(x^n) \underset{x\to 0}{=} 1 + x + x^2 + \ldots x^n + \mathrm{o}(x^n).$
$\ln(1-x) \underset{x\to 0}{=}$	$\displaystyle -\sum_{k=1}^{n} \frac{x^k}{k} + \mathrm{o}(x^n) \underset{x\to 0}{=} -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots - \frac{x^n}{n} + \mathrm{o}(x^n).$
$\dfrac{1}{1+x} \underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} (-1)^k x^k + \mathrm{o}(x^n) \underset{x\to 0}{=} 1 - x + x^2 + \ldots (-1)^n x^n + \mathrm{o}(x^n).$
$\ln(1+x) \underset{x\to 0}{=}$	$\displaystyle\sum_{k=1}^{n} (-1)^{k+1} \frac{x^k}{k} + \mathrm{o}(x^n) \underset{x\to 0}{=} x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + (-1)^{n+1} \frac{x^n}{n} + \mathrm{o}(x^n).$
$\arctan(x) \underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1} + \mathrm{o}(x^{2n+2}) \underset{x\to 0}{=} x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} + \mathrm{o}(x^{2n+2}).$
$(1+x)^\alpha \underset{x\to 0}{=}$	$\displaystyle 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \ldots + \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)}{n!} x^n + \mathrm{o}(x^n)$.
$\underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} \binom{\alpha}{k}_{\mathbb{R}} x^k + \mathrm{o}(x^n)$ où $\binom{\alpha}{k}_{\mathbb{R}}$ est le coefficient du binôme généralisé à $\alpha \in \mathbb{R}$
$\sqrt{1+x} \underset{x\to 0}{=}$	$\displaystyle 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots + (-1)^{n-1} \frac{1 \times 3 \times \ldots \times (2n-3)}{2 \times 4 \times \ldots \times 2n} x^n + \mathrm{o}(x^n).$
$\dfrac{1}{\sqrt{1+x}} \underset{x\to 0}{=}$	$\displaystyle 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \ldots + (-1)^n \frac{1 \times 3 \times \ldots \times (2n-1)}{2 \times 4 \times \ldots \times 2n} x^n + \mathrm{o}(x^n).$
$\mathrm{e}^x \underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} \frac{x^k}{k!} + \mathrm{o}(x^n) \underset{x\to 0}{=} 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots + \frac{x^n}{n!} + \mathrm{o}(x^n).$
$\cos(x) \underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} + \mathrm{o}(x^{2n+1}) \underset{x\to 0}{=} 1 - \frac{x^2}{2} + \frac{x^4}{24} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \mathrm{o}(x^{2n+1}).$
$\sin(x) \underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \mathrm{o}(x^{2n+2}) \underset{x\to 0}{=} x - \frac{x^3}{6} + \frac{x^5}{120} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \mathrm{o}(x^{2n+2}).$
$\mathrm{ch}(x) \underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} + \mathrm{o}(x^{2n+1}) \underset{x\to 0}{=} 1 + \frac{x^2}{2} + \frac{x^4}{24} + \ldots + \frac{x^{2n}}{(2n)!} + \mathrm{o}(x^{2n+1}).$
$\mathrm{sh}(x) \underset{x\to 0}{=}$	$\displaystyle\sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!} + \mathrm{o}(x^{2n+2}) \underset{x\to 0}{=} x + \frac{x^3}{6} + \frac{x^5}{120} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + \mathrm{o}(x^{2n+2}).$
$\tan(x) \underset{x\to 0}{=}$	$\displaystyle x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \mathrm{o}(x^8).$

Exercice 1 : Effectuer le DL au voisinage de 0 à l'ordre indiqué de la fonction suivante :

1. $\tan^2(x)$ $n=7$
2. $\dfrac{\ln(1+x)}{1+x}$ $n=4$
3. $e^{x\sin(x)}$ $n=6$
4. $\sqrt{1+\sin(x)}$ $n=3$
5. $\sqrt{\cos(x)}$ $n=7$
6. $\dfrac{1}{\cos(x)}$ $n=5$
7. $\dfrac{x}{\sin(x)}$ $n=5$
8. $\ln\left(\dfrac{\sin(x)}{x}\right)$ $n=5$
9. $\ln\left(\dfrac{\arctan(x)}{x}\right)$ $n=7$
10. $(1+x)^x$ $n=5$
11. $(1+x)^{\frac{1}{x}}$ $n=3$
12. $\ln\left(\ln\left((1+x)^{\frac{1}{x}}\right)\right)$ $n=4$
13. $\left(\dfrac{\sin(x)}{x}\right)^{\frac{3}{x^2}}$ $n=3$
14. $(\cos(x))^p$ $n=6$
15. $\left(\dfrac{1+e^x}{2}\right)^p$ $n=2$
16. $\left(\dfrac{\sin(x)}{x}\right)^p$ $n=5$

Correction :

1. $\tan^2(x) = x^2 + \dfrac{2}{3}x^4 + \dfrac{17}{45}x^6 + \underset{x\to 0}{o}(x^7)$

2. $\dfrac{\ln(1+x)}{1+x} = x - \dfrac{3}{2}x^2 + \dfrac{11}{6}x^3 - \dfrac{25}{12}x^4 + \underset{x\to 0}{o}(x^4)$

3. $$\begin{aligned} e^{x\sin(x)} &= e^{\overbrace{x^2-\frac{1}{6}x^4+\frac{1}{120}x^6+o(x^6)}^{o(1)}} \\ &= 1 + x^2 - \frac{1}{6}x^4 + \frac{1}{120}x^6 + o\left(x^6\right) + \frac{1}{2}\left(x^2 - \frac{1}{6}x^4 + o\left(x^4\right)\right)^2 + \frac{1}{6}\left(x^2 + o\left(x^2\right)\right)^3 \\ &= 1 + x^2 + \left(-\frac{1}{6} + \frac{1}{2}\right)x^4 + \left(\frac{1}{120} - \frac{1}{6} + \frac{1}{6}\right)x^6 + o\left(x^6\right) \\ &= 1 + x^2 + \frac{1}{3}x^4 + \frac{1}{120}x^6 + \underset{x\to 0}{o}(x^6) \end{aligned}$$

4. $\sqrt{1+\sin(x)} = 1 + \dfrac{1}{2}x - \dfrac{1}{8}x^2 - \dfrac{1}{48}x^3 + \underset{x\to 0}{o}(x^3)$

5. $\sqrt{\cos(x)} = 1 - \dfrac{1}{4}x^2 - \dfrac{1}{96}x^4 - \dfrac{19}{5760}x^6 + \underset{x\to 0}{o}(x^7)$

6. $\dfrac{1}{\cos(x)} = 1 + \dfrac{1}{2}x^2 + \dfrac{5}{24}x^4 + \underset{x\to 0}{o}(x^5)$

7. $\dfrac{x}{\sin(x)} = 1 + \dfrac{1}{6}x^2 + \dfrac{7}{360}x^4 + \underset{x\to 0}{o}(x^5)$

8. $\ln\left(\dfrac{\sin(x)}{x}\right) = -\dfrac{1}{6}x^2 - \dfrac{1}{180}x^4 + \underset{x\to 0}{o}(x^5)$

9. $\ln\left(\dfrac{\arctan(x)}{x}\right) = -\dfrac{1}{3}x^2 + \dfrac{13}{90}x^4 - \dfrac{251}{2835}x^6 + \underset{x\to 0}{o}(x^7)$

10. $(1+x)^x = 1 + x^2 - \dfrac{1}{2}x^3 + \dfrac{5}{6}x^4 - \dfrac{3}{4}x^5 + \underset{x\to 0}{o}(x^5)$

11. $(1+x)^{\frac{1}{x}} = e \times \left[1 - \frac{1}{2}\,x + \frac{11}{24}\,x^2 - \frac{7}{16}\,x^3\right] + \underset{x\to 0}{\mathrm{o}}(x^3)$

12. $\ln\left(\ln\left((1+x)^{\frac{1}{x}}\right)\right) = -\frac{1}{2}\,x + \frac{5}{24}\,x^2 - \frac{1}{8}\,x^3 + \frac{251}{2880}\,x^4 + \underset{x\to 0}{\mathrm{o}}(x^4)$

13. $\left(\frac{\sin(x)}{x}\right)^{\frac{3}{x^2}} = \frac{1}{\sqrt{e}} \times \left[1 - \frac{1}{60}\,x^2\right] + \underset{x\to 0}{\mathrm{o}}(x^3)$

14. $(\cos(x))^p = 1 - \frac{p}{2}x^2 + \frac{p(3p-2)}{24}x^4 - \frac{p[15(p-1)^2+1]}{720}x^6 + \underset{x\to 0}{\mathrm{o}}(x^6)$

15. $\left(\frac{1+e^x}{2}\right)^p = 1 + \frac{p}{2}\,x + \frac{p\,(p+1)}{8}\,x^2 + \underset{x\to 0}{\mathrm{o}}(x^2)$

16. $\left(\frac{\sin(x)}{x}\right)^p = 1 - \frac{p}{6}\,x^2 + \frac{p\,(5p-2)}{360}\,x^4 + \underset{x\to 0}{\mathrm{o}}(x^5)$

Exercice 2 (Applications) : Calculer les limites suivantes :

1. $\lim\limits_{x\to 0} \frac{\sin(\sin(x)-x)}{\sqrt{1+x^3}-1}$
2. $\lim\limits_{x\to 0} \frac{3\tan 4x - 4\tan 3x}{3\sin 4x - 4\sin 3x}$
3. $\lim\limits_{x\to 0} \frac{x(e^x+1)-2(e^x-1)}{x^3}$
4. $\lim\limits_{x\to 0} \frac{\ln(1+x)-\ln(1-x)}{\arctan(1+x)-\arctan(1-x)}$
5. $\lim\limits_{x\to 0} \left(\frac{1}{x} - \frac{1}{\ln(1+x)}\right)$
6. $\lim\limits_{x\to 0} \frac{(1+x)^{\frac{1}{x}}-e}{x}$
7. $\lim\limits_{x\to 1} \frac{1-x+\ln(x)}{1-\sqrt{2x-x^2}}$
8. $\lim\limits_{x\to 1} \left(\frac{1}{2(1-\sqrt{x})} - \frac{1}{3(1-\sqrt[3]{x})}\right)$

Correction :

1. ◇ $\sin(x) - x \underset{x\to 0}{\longrightarrow} 0$ donc $\sin(\sin(x)-x) \underset{x\to 0}{\sim} \sin(x) - x$.

$$\sin(x) - x = \left[x - \frac{x^3}{3!} + \underset{x\to 0}{\mathrm{o}}(x^3)\right] - x = -\frac{x^3}{3!} + \underset{x\to 0}{\mathrm{o}}(x^3) \underset{x\to 0}{\sim} -\frac{x^3}{3!}, \text{ d'où}$$

$$\sin(\sin(x)-x) \underset{x\to 0}{\sim} -\frac{x^3}{3!}$$

◇ Comme $x^3 \underset{x\to 0}{\longrightarrow} 0$, $\sqrt{1+x^3} - 1 \underset{x\to 0}{\sim} \frac{1}{2}x^3$.

Donc $\frac{\sin(\sin(x)-x)}{\sqrt{1+x^3}-1} \underset{x\to 0}{\sim} \frac{-\frac{x^3}{3!}}{\frac{1}{2}x^3} \underset{x\to 0}{\sim} -\frac{1}{3}$ et $\lim\limits_{x\to 0} \frac{\sin(\sin(x)-x)}{\sqrt{1+x^3}-1} = -\frac{1}{3}$.

2. ◇ $3\tan 4x - 4\tan 3x = 3\left[4x + \frac{(4x)^3}{3} + \underset{x\to 0}{\mathrm{o}}(x^3)\right] - 4\left[3x + \frac{(3x)^3}{3} + \underset{x\to 0}{\mathrm{o}}(x^3)\right]$

$$= 64x^3 - 36x^3 + \underset{x\to 0}{\mathrm{o}}(x^3) \underset{x\to 0}{\sim} 28x^3$$

$\diamond$ $$3\sin 4x - 4\sin 3x = 3\left[4x - \frac{(4x)^3}{3!} + \underset{x\to 0}{\mathrm{o}}\left(x^3\right)\right] - 4\left[3x - \frac{(3x)^3}{3!} + \underset{x\to 0}{\mathrm{o}}\left(x^3\right)\right]$$
$$= -32x^3 + 18x^3 + \underset{x\to 0}{\mathrm{o}}\left(x^3\right) \underset{x\to 0}{\sim} -14x^3$$

Donc $\dfrac{3\tan 4x - 4\tan 3x}{3\sin 4x - 4\sin 3x} \underset{x\to 0}{\sim} \dfrac{28x^3}{-14x^3} \underset{x\to 0}{\sim} -2$ et $\displaystyle\lim_{x\to 0} \frac{3\tan 4x - 4\tan 3x}{3\sin 4x - 4\sin 3x} = -2.$

3. $$\frac{x(e^x+1) - 2(e^x-1)}{x^3} = \frac{x\left[1 + x + \frac{x^2}{2} + \underset{x\to 0}{\mathrm{o}}\left(x^2\right) + 1\right] - 2\left[1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \underset{x\to 0}{\mathrm{o}}\left(x^3\right) - 1\right]}{x^3}$$
$$= \frac{\left[2x + x^2 + \frac{x^3}{2} + \underset{x\to 0}{\mathrm{o}}\left(x^3\right)\right] - \left[2x + x^2 + \frac{x^3}{3} + \underset{x\to 0}{\mathrm{o}}\left(x^3\right)\right]}{x^3}$$
$$= \frac{\frac{1}{6}x^3 + \underset{x\to 0}{\mathrm{o}}\left(x^3\right)}{x^3} = \frac{1}{6} + \underset{x\to 0}{\mathrm{o}}(1) \quad \text{et} \quad \lim_{x\to 0} \frac{x(e^x+1) - 2(e^x-1)}{x^3} = \frac{1}{6}.$$

4. $\diamond$ $\ln(1+x) - \ln(1-x) = \left[x + \underset{x\to 0}{\mathrm{o}}(x)\right] - \left[-x + \underset{x\to 0}{\mathrm{o}}(x)\right] = 2x + \underset{x\to 0}{\mathrm{o}}(x) \underset{x\to 0}{\sim} 2x.$

$\diamond$ $f(x) = \arctan(1+x)$. On a $f'(x) = \dfrac{1}{1+(1+x)^2}$ donc $f'(0) = \dfrac{1}{2}$.

Le $\mathrm{DL}_1(0)$ de f est donné par $f(x) = f(0) + f'(0)x + \underset{x\to 0}{\mathrm{o}}(x)$

$$\begin{cases} \arctan(1+x) = \dfrac{\pi}{4} + \dfrac{1}{2}x + \underset{x\to 0}{\mathrm{o}}(x) \\ \arctan(1-x) = \dfrac{\pi}{4} - \dfrac{1}{2}x + \underset{x\to 0}{\mathrm{o}}(x) \end{cases}$$

Donc $\arctan(1+x) - \arctan(1-x) = x + \underset{x\to 0}{\mathrm{o}}(x) \underset{x\to 0}{\sim} x.$

D'où,
$$\lim_{x\to 0} \frac{\ln(1+x) - \ln(1-x)}{\arctan(1+x) - \arctan(1-x)} = 2.$$

5. $$\frac{1}{x} - \frac{1}{\ln(1+x)} = \frac{\ln(1+x) - x}{x\ln(1+x)}$$
$$= \frac{\left[x - \frac{x^2}{2} + \underset{x\to 0}{\mathrm{o}}\left(x^2\right)\right] - x}{x\ln(1+x)} = \frac{-\frac{x^2}{2} + \underset{x\to 0}{\mathrm{o}}\left(x^2\right)}{x\ln(1+x)}$$
$$\underset{x\to 0}{\sim} \frac{-\frac{x^2}{2}}{x^2} \underset{x\to 0}{\sim} -\frac{1}{2} \quad \text{et} \quad \lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\ln(1+x)}\right) = -\frac{1}{2}.$$

6. $$(1+x)^{\frac{1}{x}} = e^{\frac{1}{x}\ln(1+x)} = e^{\frac{1}{x}\left[x - \frac{x^2}{2} + \underset{x\to 0}{\mathrm{o}}\left(x^2\right)\right]} = e^{1 - \frac{x}{2} + \underset{x\to 0}{\mathrm{o}}(x)} = e \times e^{-\frac{x}{2} + \underset{x\to 0}{\mathrm{o}}(x)}$$
$$= e\left[1 - \frac{x}{2} + \underset{x\to 0}{\mathrm{o}}(x)\right] = e - \frac{e}{2}x + \underset{x\to 0}{\mathrm{o}}(x).$$

Donc, $\dfrac{(1+x)^{\frac{1}{x}} - e}{x} = \dfrac{e - \frac{e}{2}x + \underset{x\to 0}{\mathrm{o}}(x) - e}{x} = -\dfrac{e}{2} + \underset{x\to 0}{\mathrm{o}}(1)$ et $\displaystyle\lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x} = -\frac{e}{2}.$

7. On pose $x = 1 + h$.

$$\frac{1 - x + \ln(x)}{1 - \sqrt{2x - x^2}} = \frac{1 - (1+h) + \ln(1+h)}{1 - \sqrt{2(1+h) - (1+h)^2}} = \frac{-h + \left[h - \frac{h^2}{2} + \underset{h\to 0}{\mathrm{o}}\left(h^2\right)\right]}{1 - \sqrt{1 - h^2}} \underset{h\to 0}{\sim} \frac{-\frac{h^2}{2}}{\frac{1}{2}h^2} \underset{h\to 0}{\sim} -1.$$

Donc,

$$\lim_{x\to 1}\frac{1-x+\ln(x)}{1-\sqrt{2x-x^2}}=-1.$$

8.

$$\frac{1}{2(1-\sqrt{x})}-\frac{1}{3(1-\sqrt[3]{x})}=\frac{3(1-\sqrt[3]{x})-2(1-\sqrt{x})}{6(1-\sqrt{x})(1-\sqrt[3]{x})}$$

On pose $x=1+h$.

- $3(1-\sqrt[3]{x})-2(1-\sqrt{x}) = 3\left[1-\left(1+\frac{1}{3}h-\frac{1}{9}h^2+\underset{h\to 0}{\mathrm{o}}\left(h^2\right)\right)\right]-2\left[1-\left(1+\frac{1}{2}h-\frac{1}{8}h^2+\underset{h\to 0}{\mathrm{o}}\left(h^2\right)\right)\right]$
  $= \frac{1}{12}h^2+\underset{h\to 0}{\mathrm{o}}\left(h^2\right) \underset{h\to 0}{\sim} \frac{1}{12}h^2.$
- et $6(1-\sqrt{x})(1-\sqrt[3]{x}) = 6\left(1-\sqrt{1+h}\right)\left(1-\sqrt[3]{1+h}\right) \underset{h\to 0}{\sim} 6\left(\frac{-1}{2}h\right)\left(\frac{-1}{3}h\right) \underset{h\to 0}{\sim} h^2.$

Donc,

$$\frac{1}{2(1-\sqrt{x})}-\frac{1}{3(1-\sqrt[3]{x})} \underset{h\to 0}{\sim} \frac{\frac{1}{12}h^2}{h^2} \underset{h\to 0}{\sim} \frac{1}{12} \quad \text{et} \quad \lim_{x\to 1}\left(\frac{1}{2(1-\sqrt{x})}-\frac{1}{3(1-\sqrt[3]{x})}\right)=\frac{1}{12}.$$

Développements usuels en 0