Correction du devoir maison n° 1

Exercice 1 $\quad x_n = \cos\left(\frac{\pi}{2^n}\right)$.

$x_{n+1} = \cos\left(\frac{\pi}{2^{n+1}}\right) = \cos\left(\frac{\theta}{2}\right)$ avec $\theta = \frac{\pi}{2^n}$

Or $\cos(\theta) = 2\cos^2\left(\frac{\theta}{2}\right) - 1$ d'où $\cos^2\left(\frac{\theta}{2}\right) = \dfrac{\cos\theta + 1}{2}$

i.e $x_{n+1}^2 = \dfrac{x_n + 1}{2}$.

On en déduit que $\cos\left(\frac{\pi}{8}\right) = \sqrt{\dfrac{\cos(\frac{\pi}{4} + 1}{2}} = \sqrt{\dfrac{\frac{\sqrt{2}}{2} + 1}{2}} = \sqrt{\dfrac{\sqrt{2}+2}{4}} = \dfrac{\sqrt{\sqrt{2}+2}}{2}$

On recommence ce procédé pour calculer les autres valeurs demandées.

Exercice 2

1. $\alpha = \dfrac{115\pi}{6} = \dfrac{120\pi}{6} - \dfrac{5\pi}{6} = -\dfrac{5\pi}{6}\ [2\pi]$ donc $\boxed{\cos(\alpha) = -\dfrac{\sqrt{3}}{2} \text{ et } \sin(\alpha) = -\dfrac{1}{2}}$

   $\beta = -\dfrac{55\pi}{8} = -\dfrac{48\pi}{8} - \dfrac{7\pi}{8} = -\pi + \dfrac{\pi}{8}\ [2\pi]$ donc $\cos(\beta) = -\cos\left(\dfrac{\pi}{8}\right)$ et $\sin(\beta) = -\sin\left(\dfrac{\pi}{8}\right)$

   $\cos^2\left(\dfrac{\pi}{8}\right) = \dfrac{1 + \cos\left(\frac{\pi}{4}\right)}{2} = \dfrac{1 + \frac{\sqrt{2}}{2}}{2} = \dfrac{2+\sqrt{2}}{4}$ et $\cos\left(\dfrac{\pi}{8}\right) > 0$ donc $\cos\left(\dfrac{\pi}{8}\right) = \dfrac{\sqrt{2+\sqrt{2}}}{2}$

   $\sin\left(\dfrac{\pi}{4}\right) = \cos\left(\dfrac{\pi}{8}\right)\sin\left(\dfrac{\pi}{8}\right)$ donc $\sin\left(\dfrac{\pi}{8}\right) = \dfrac{\sin\left(\frac{\pi}{4}\right)}{2\cos\left(\frac{\pi}{8}\right)} = \dfrac{\frac{\sqrt{2}}{2}}{\sqrt{2+\sqrt{2}}} = \dfrac{\sqrt{2}}{2\sqrt{2+\sqrt{2}}}$

   Finalement, $\boxed{\cos(\beta) = -\dfrac{\sqrt{2+\sqrt{2}}}{2} \text{ et } \sin(\beta) = -\dfrac{\sqrt{2}}{2\sqrt{2+\sqrt{2}}}}$

   $\gamma = \dfrac{101\pi}{12} = \dfrac{96\pi}{12} + \dfrac{5\pi}{12} = \dfrac{\pi}{2} - \dfrac{\pi}{12}\ [2\pi]$ donc $\cos(\gamma) = \sin\left(\dfrac{\pi}{12}\right)$ et $\sin(\gamma) = \cos\left(\dfrac{\pi}{12}\right)$
   Par la même démarche que celle pour trouver le cosinus et le sinus de $\dfrac{\pi}{8}$, on obtient

   $\boxed{\cos(\gamma) = \dfrac{1}{2\sqrt{2+\sqrt{3}}} \text{ et } \sin(\gamma) = \dfrac{\sqrt{2+\sqrt{3}}}{2}}$

2. a) $\cos\left(2x - \dfrac{\pi}{3}\right) = -\dfrac{1}{2} = \cos\left(\dfrac{2\pi}{3}\right) \Leftrightarrow 2x - \dfrac{\pi}{3} = \dfrac{2\pi}{3}\ [2\pi]$ ou $2x - \dfrac{\pi}{3} = -\dfrac{2\pi}{3}\ [2\pi]$

   $\boxed{\Leftrightarrow x = \dfrac{\pi}{2}\ [\pi] \text{ ou } x = -\dfrac{\pi}{6}\ [\pi]}$

b) $\cos\left(x+\dfrac{3\pi}{4}\right) = \sin\left(\dfrac{x}{4}\right) = \cos\left(\dfrac{\pi}{2} - \dfrac{x}{4}\right)$

$\Leftrightarrow x + \dfrac{3\pi}{4} = \dfrac{\pi}{2} - -\dfrac{x}{4}\,[2\pi]$ ou $x - \dfrac{3\pi}{4} = -\dfrac{\pi}{2} + -\dfrac{x}{4}\,[2\pi]$

$\Leftrightarrow \dfrac{5}{4}x = -\dfrac{\pi}{4}\,[2\pi]$ ou $\dfrac{3}{4}x = -\dfrac{5\pi}{4}\,[2\pi]$ $\boxed{\Leftrightarrow x = -\dfrac{\pi}{5}\,[\dfrac{8\pi}{5}] \text{ ou } x = -\dfrac{5\pi}{3}\,[\dfrac{8\pi}{5}]}$

3. a) $\sin\left(\dfrac{x}{2}\right) > \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2} \Leftrightarrow \dfrac{x}{2} \in]\dfrac{\pi}{4} + 2k\pi\ ;\ \dfrac{3\pi}{4} + 2k\pi[, k \in \mathbb{Z}$

$\boxed{\Leftrightarrow x \in]\dfrac{\pi}{2} + 4k\pi\ ;\ \dfrac{3\pi}{2} + 4k\pi[, k \in \mathbb{Z}}$

b) **Attention aux valeurs interdites !** Ici $2x \neq \dfrac{\pi}{2} + k\pi \Leftrightarrow x \neq \dfrac{\pi}{4} + \dfrac{k\pi}{2}, k \in \mathbb{Z}$

$\tan(2x) \geq -1 = \tan\left(-\dfrac{\pi}{4}\right) \Leftrightarrow 2x \in [-\dfrac{\pi}{4} + 2k\pi\ ;\ \dfrac{\pi}{2} + 2k\pi[\cup[\dfrac{3\pi}{4} + 2k\pi\ ;\ \dfrac{3\pi}{2} + 2k\pi[, k \in \mathbb{Z}$

$\boxed{\Leftrightarrow x \in [-\dfrac{\pi}{8} + k\pi\ ;\ \dfrac{\pi}{4} + k\pi[\cup[\dfrac{3\pi}{8} + k\pi\ ;\ \dfrac{3\pi}{4} + k\pi[, k \in \mathbb{Z}}$