Correction du DS n° 3

PTSI 25/26

Exo 1

$\theta \neq 0 [\pi]$

Partie 1

$$A_n(\theta) = \sum_{k=0}^{n} \cos^2(k\theta)$$

$$B_n(\theta) = \sum_{k=0}^{n} \sin^2(k\theta)$$

1) $A_n(\theta) + B_n(\theta) = \sum_{k=0}^{n} \underbrace{\cos^2(k\theta) + \sin^2(k\theta)}_{1}$
par linéarité

$= n+1$

$A_n(\theta) - B_n(\theta) = \sum_{k=0}^{n} \cos^2(k\theta) - \sin^2(k\theta)$
par linéarité

$$= \sum_{k=0}^{n} \cos(2k\theta)$$

2) $\sum_{k=0}^{n} e^{2ik\theta} = \dfrac{1 - e^{2i(n+1)\theta}}{1 - e^{2i\theta}}$

$e^{2i\theta} \neq 1$
car $\theta \neq 0 [\pi]$

$$= \frac{e^{i(n+1)\theta}(-2i\sin(n+1)\theta)}{e^{i\theta}(-2i\sin\theta)}$$

$$\boxed{= e^{ni\theta} \frac{\sin(n+1)\theta}{\sin\theta}}$$

3) $A_n(\theta) - B_n(\theta) = \mathrm{Re}\left(\sum_{k=0}^{n} e^{2ik\theta}\right)$

$$= \frac{\cos(n\theta)\sin(n+1)\theta}{\sin(\theta)}$$

4)
$$\boxed{\begin{aligned} A_n(\theta) &= \frac{1}{2}\left(n+1+\frac{\cos(n\theta)\sin(n+1)\theta}{\sin\theta}\right) \\ B_n(\theta) &= \frac{1}{2}\left(n+1-\frac{\cos(n\theta)\sin(n+1)\theta}{\sin\theta}\right) \end{aligned}}$$

Partie 2

$$S_n = \sum_{1 \leq i,j \leq n} |i-j|$$

0) $S_1 = 0 \quad S_2 = 2 \quad S_3 = 8 \quad S_4 = 20$

1) $S_n = \sum_{i=1}^{n} \sum_{j=1}^{n} |i-j|$

$$= \sum_{i=1}^{n} 2 \sum_{j=1}^{i-1} (i-j)$$

$|i-j| = 0$ si $i=j$
et $|i-j| = |j-i| = i-j$ si $j<i$

$$= \sum_{i=1}^{n} 2\left(\underbrace{\sum_{j=1}^{i-1} i}_{i(i-1)} - \underbrace{\sum_{j=1}^{i-1} j}_{\frac{i(i-1)}{2}}\right)$$

$$= \sum_{i=1}^{n} i(i-1)$$

$$= \sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i$$

$$= \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{2n+1-3}{3}\right]$$

$$\boxed{= \frac{n(n+1)(n-1)}{3}}$$

Exo 2

0) $y'' + 2\alpha y' + (1+\alpha^2)y = 0$

(E_c) $r^2 + 2\alpha r + (1+\alpha^2) = 0$

$\Delta = 4\alpha^2 - 4(1+\alpha^2) = -4$

$r_1 = -\alpha + i \qquad r_2 = -\alpha - i$

$$\boxed{\begin{gathered} y_H(x) = e^{-\alpha x}(A\cos x + B\sin x) \\ A, B, x \in \mathbb{R} \end{gathered}}$$

1) $e^{-\alpha x}\cos x = \mathrm{Re}\left(e^{(-\alpha+i)x}\right)$

Or $-\alpha+i$ est racine simple de (E_c)

$y_p = k x e^{(-\alpha+i)x}$

$y'_p = k e^{(-\alpha+i)x} + kx(-\alpha+i)e^{(-\alpha+i)x}$

$= k e^{(-\alpha+i)x}(1+x(-\alpha+i))$

$y''_p = k(-\alpha+i)(1+x(-\alpha+i))e^{(-\alpha+i)x} + k(-\alpha+i)e^{(-\alpha+i)x}$

$= k e^{(-\alpha+i)x}\left(2(-\alpha+i)+x(-\alpha+i)^2\right)$

$y''_p + 2\alpha y'_p + (1+\alpha^2)y_p = e^{(-\alpha+i)x}$

$\Leftrightarrow k\left(2(-\alpha+i)+x(-\alpha+i)^2\right) + 2\alpha k\left(1+x(-\alpha+i)\right) + (1+\alpha^2)kx = 1$

$\Leftrightarrow x\left(\underbrace{(-\alpha+i)^2}_{\alpha^2-2i\alpha-1} - 2\alpha^2 + 2i\alpha + 1 + \alpha^2\right)k + k\left(2(-\alpha+i)+2\alpha\right) = 1$

$\Leftrightarrow k = \dfrac{1}{2i}$ et $y_p = -\dfrac{i}{2}x e^{(-\alpha+i)x}$

puis $z_p = \mathrm{Re}(y_p)$

$= \dfrac{x e^{-\alpha x}}{2}\sin x$

2) $\boxed{y(x) = e^{-\alpha x}\left(A\cos x + B\sin x + \dfrac{x}{2}\sin x\right) \quad A, B, x \in \mathbb{R}}$

3) $y(0)=0$

$\Leftrightarrow A=0$

et $y'(x) = e^{-\alpha x}\left(-A\sin x + B\cos x + \dfrac{1}{2}\sin x + \dfrac{x}{2}\cos x\right) - \alpha e^{-\alpha x}\left(A\cos x + B\sin x + \dfrac{x}{2}\sin x\right)$

$y'(0)=0 \Leftrightarrow B=0$

$y(x) = \dfrac{x}{2}\sin x\, e^{-\alpha x}$

Exo 3

0) $y' + \sin x\, y = 0$

$y_H(x) = Ce^{+\cos x}, \quad C, x\in\mathbb{R}$

1) $\int e^{-\cos x}\sin x\cos x\,dx$

$= \int -u e^{-u}\,du = u e^{-u} - \int e^{-u}du$

$= u e^{-u} + e^{-u}$

$u = \cos x$

$du = -\sin x\,dx$

$= \cos x\, e^{-\cos x} + e^{-\cos x}$

2) On pose $y_p(x) = Ce^{+\cos x}$

$C' = \sin(2x)e^{-\cos x}$

$= 2\sin x\cos x\, e^{-\cos x}$

$C = 2(\cos x + 1)e^{-\cos x}$

et $y_p = 2(\cos x+1)$

$y = Ce^{\cos x} + 2(\cos x+1) \quad C, x\in\mathbb{R}$

3) $y\left(\frac{\pi}{2}\right) = 0 \Leftrightarrow C+2=0$

$\Leftrightarrow C=-2$

$\boxed{y(x) = 2e^{\cos x} - 2(\cos x+1)}$

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Exercice n° 4

$$\begin{cases} x + y + \lambda z = 1 \\ \lambda x + y + z = 1 \\ 3x + (\lambda+2)y + (2\lambda+1)z = 3 \end{cases} \iff \begin{cases} x + y + \lambda z = 1 \\ (1-\lambda)y + (1-\lambda^2)z = 1-\lambda \\ (\lambda-1)y + (1-\lambda)z = 0 \end{cases}$$

$L_2 \leftarrow L_2 - \lambda L_1$

$L_3 \leftarrow L_3 - 3L_1$

$$\iff \begin{cases} x + y + \lambda z = 1 \\ (\lambda-1)y + (1-\lambda)z = 0 \\ \underbrace{(2-\lambda-\lambda^2)}_{(1-\lambda)(2+\lambda)}z = 1-\lambda \end{cases}$$

$L_2 \leftarrow L_3$

$L_3 \leftarrow L_3 + L_2$

Pour $\lambda = 1$ on obtient $x+y+z=1$ 1 pivot

Pour $\lambda = -2$ le système est incompatible car $L_3 \iff 0 = 3$ 2 pivots

Pour $\lambda \neq 1$ et $\lambda \neq -2$ on obtient 3 pivots

$$\begin{cases} z = \dfrac{1-\lambda}{2-\lambda-\lambda^2} = \dfrac{1}{2+\lambda} \\ y = z = \dfrac{1}{2+\lambda} \\ x = 1 - y - \lambda z \\ \quad = 1 - \dfrac{1+\lambda}{2+\lambda} = \dfrac{1}{2+\lambda} \end{cases}$$

Pour $\lambda = 1$

$S = \{(x,y,z) \in \mathbb{R}^3 / x+y+z=1\}$

Pour $\lambda = -2$ $S = \emptyset$

Pour $\lambda \neq 1$ et $\lambda \neq -2$

$S = \left\{ \dfrac{1}{2+\lambda}(1,1,1) \right\}$