Démos $\vec{V}$ et $\vec{a}$ en coordonnées cylindriques

$$\boxed{\overrightarrow{OM} = r.\vec{u}_r + z\,\vec{u}_z}$$

$$\vec{V} = \frac{d\overrightarrow{OM}}{dt} = \frac{d}{dt}\left(r\vec{u}_r + z\vec{u}_z\right)$$

$$= \dot{r}.\vec{u}_r + r\frac{d\vec{u}_r}{dt} + \dot{z}\,\vec{u}_z + z\frac{d\vec{u}_z}{dt}$$

Remarques : $\frac{d\vec{u}_r}{dt} = \frac{d\vec{u}_r}{d\theta}.\frac{d\theta}{dt} = \underbrace{\frac{d\vec{u}_r}{d\theta}}_{\vec{u}_\theta}.\dot{\theta} = \dot{\theta}\,\vec{u}_\theta$

et $\vec{u}_z$ est fixe $\Rightarrow \frac{d\vec{u}_z}{dt} = \vec{0}$

Donc $\boxed{\vec{V} = \dot{r}\,\vec{u}_r + r\dot{\theta}\,\vec{u}_\theta + \dot{z}\,\vec{u}_z} = \begin{pmatrix}\dot{r}\\ r\dot{\theta}\\ \dot{z}\end{pmatrix}$

$$\vec{a} = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{d\vec{V}}{dt} = \frac{d}{dt}\left(\dot{r}\vec{u}_r + r\dot{\theta}\vec{u}_\theta + \dot{z}\vec{u}_z\right)$$

$$\vec{a} = \underbrace{\frac{d}{dt}\left(\dot{r}\vec{u}_r\right)} + \frac{d}{dt}\left(r.\dot{\theta}.\vec{u}_\theta\right) + \frac{d}{dt}\left(\dot{z}\vec{u}_z\right)$$

$$= \ddot{r}\vec{u}_r + \dot{r}\dot{\theta}\vec{u}_\theta + \dot{r}\dot{\theta}\vec{u}_\theta + r\ddot{\theta}\vec{u}_\theta + r.\dot{\theta}\frac{d}{dt}\vec{u}_\theta + \ddot{z}\vec{u}_z$$

Remarque : $\frac{d\vec{u}_\theta}{dt} = \frac{d\vec{u}_\theta}{d\theta}\times\frac{d\theta}{dt} = \frac{d\vec{u}_\theta}{d\theta}\times\dot{\theta} = -\vec{u}_r.\dot{\theta} = \boxed{-\dot{\theta}\,\vec{u}_r}$

On regroupe les termes devant $\vec{u}_r$, $\vec{u}_\theta$ et $\vec{u}_z$:

$$\vec{a} = \left(\ddot{r} - r.\dot{\theta}.\dot{\theta}\right)\vec{u}_r + \left(\dot{r}\dot{\theta} + \dot{r}\dot{\theta} + r\ddot{\theta}\right)\vec{u}_\theta + \ddot{z}\,\vec{u}_z$$

$$\boxed{\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{u}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\vec{u}_\theta + \ddot{z}\,\vec{u}_z}$$